Secondary Mathematics 3 Table of Contents

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Polynomials

Unit 2 Cluster 4 (A.APR.1): Polynomials

Cluster 4: Perform arithmetic operations on polynomials
- 2.4 Polynomials are closed under addition, subtraction, and multiplication
- 2.4 Add, subtract, and multiply polynomials (**NO DIVISION**)

VOCABULARY

A term that does not have a variable is called a **constant.** For example the number 5 is a **constant** because it does not have a variable attached and will always have the value of 5.

A constant or a variable, or a product of a constant and a variable is called a **term**. For example 2, x, or $-3x^2$ are all terms.

Terms with the same variable to the same power are **like terms**. $2x^2$ and $-7x^2$ are like terms.

An expression formed by adding a finite number of unlike terms is called a **polynomial**. The variables can only be raised to positive integer exponents. $4x^3-6x^2+1$ is a polynomial, while $x^{\frac{3}{2}}-2x^{-1}+5$ is not a polynomial. **NOTE:** There are no square roots of variables, no fractional powers, and no variables in the denominator of any fractions.

A polynomial with only one term is called a **monomial** $\left(6x^4\right)$. A polynomial with two terms is called a **binomial** $\left(2x+1\right)$. A polynomial with three terms is called a **trinomial** $\left(5x^2-x+3\right)$.

Polynomials are in **standard (general) form** when written with exponents in descending order and the constant term last. For example $2x^4-5x^3+7x^2-x+3$ is in standard form.

The exponent of a term gives you the **degree** of the term. The term $-3x^2$ has degree two. For a polynomial, the value of the *largest exponent* is the **degree** of the whole polynomial. The polynomial $2x^4-5x^3+7x^2-x+3$ has degree 4.

When the term contains a variable and a number, the number part of a term is called the **coefficient**. $6x$ has a coefficient of 6 and $-x^2$ has a coefficient of -1.

The **leading coefficient** is the coefficient of the first term when the polynomial is written in standard form. 2 is the leading coefficient of $2x^4-5x^3+7x^2-x+3$.

Degree n

General Polynomial: $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$

Leading Coefficient a_n

Leading Term

Constant

CLASSIFICATIONS OF POLYNOMIALS

Name	Form	Degree	Example
Zero	$f(x)=0$	None	$f(x)=0$
Constant	$f(x)=a, a\neq 0$	0	$f(x)=5$
Linear	$f(x)=ax+b$	1	$f(x)=-2x+1$
Quadratic	$f(x)=ax^2+bx+c$	2	$f(x)=3x^2+\frac{1}{2}x+\frac{7}{9}$
Cubic	$f(x)=ax^3+bx^2+cx+d$	3	$f(x)=x^3-3x^2$

Polynomial Operations

Addition/Subtraction: Combine like terms.

Example 1:

Horizontal Method	Vertical Method
$(2x^3-3x^2+4x-1)+(x^3+2x^2-5x+3)$ $(2x^3+x^3)+(-3x^2+2x^2)+(4x-5x)+(-1+3)$ $3x^3-x^2-x+2$	$\begin{array}{r} 2x^3-3x^2+4x-1 \\ +\quad x^3+2x^2-5x+3 \\ \hline 3x^3-\ x^2-\ x+2 \end{array}$

Example 2:

Horizontal Method	Vertical Method
$(4x^2+3x-4)-(2x^3+x^2-x+2)$ $4x^2+3x-4-2x^3-x^2+x-2$ $-2x^3+3x^2+4x-6$	$\begin{array}{r} 4x^2+3x-4 \\ -(2x^3+x^2-\ x+2) \\ \hline -2x^3+3x^2+4x-6 \end{array}$

Practice Exercises A

Perform the required operations. Write your answers in standard form and determine if the result is a polynomial.

1. $(3x^2-4x+1)+(-x^2+x-9)$
2. $(5x^2+x^3+6)+(x^2+5-6x)$
3. $(x^2+1)+(-4x^2+5)$
4. $(5x^2-4x+1)-(8-x^2)$
5. $(-3+4n^2)-(5-2n^3)$
6. $(3t^2-8t+2)-(-3t^2+5t-7)$

7. $\left(7+2x-4x^2\right)+\left(-3x+x^2-5\right)$

8. $\left(-8x^2-3x+7\right)+\left(-x^3+6x^2-5\right)$

9. $\left(9x^3-5x^2+x\right)+\left(6x^2+5x-10\right)$

10. $12-\left(-5x^2+x-7\right)$

11. $\left(x-4x^2+7\right)-\left(-5x^2+5x-3\right)$

12. $\left(3x^2+4\right)-\left(x^2-5x+2\right)$

Multiplication: Multiply two binomials $(5x-7)(2x+9)$

<table>
<tr><th>Distributive (FOIL) Method</th><th>Box Method</th><th>Vertical Method</th></tr>
<tr>
<td>$(5x-7)(2x+9)$
$5x(2x+9)-7(2x+9)$
$10x^2+45x-14x-63$
combine like terms
$10x^2+31x-63$</td>
<td>

	$5x$	-7
$2x$	$10x^2$	$-14x$
9	$45x$	-63

**combine terms on the diagonals of the unshaded boxes(top right to lower left)*

$10x^2+31x-63$</td>
<td>$$\begin{array}{lr} & 5x-7 \\ \times & 2x+9 \\ \hline & 45x-63 \\ 10x^2-14x & \\ \hline 10x^2+31x-63 & \end{array}$$</td>
</tr>
</table>

Multiplication: Multiply a binomial and a trinomial $(2x+3)\left(6x^2-7x-5\right)$

<table>
<tr><th>Distributive Method</th><th>Box Method</th><th>Vertical Method</th></tr>
<tr>
<td>$(2x+3)\left(6x^2-7x-5\right)$
$2x\left(6x^2-7x-5\right)+3\left(6x^2-7x-5\right)$
$\left(12x^3-14x^2-10x\right)+\left(18x^2-21x-15\right)$
$12x^3-14x^2-10x+18x^2-21x-15$
combine like terms
$12x^3+4x^2-31x-15$</td>
<td>

	$6x^2$	$-7x$	-5
$2x$	$12x^3$	$-14x^2$	$-10x$
3	$18x^2$	$-21x$	-15

**combine terms on the diagonals of the unshaded boxes(top right to lower left)*

$12x^3+4x^2-31x-15$</td>
<td>$$\begin{array}{lr} & 6x^2-7x-\ 5 \\ \times & 2x+\ 3 \\ \hline & 18x^2-21x-15 \\ 12x^3-14x^2-10x & \\ \hline 12x^3+\ 4x^2-31x-15 & \end{array}$$</td>
</tr>
</table>

Multiplication: Multiply a trinomial and a trinomial $\left(2x^2+3x-1\right)\left(6x^2-7x-5\right)$

Distributive Method	Box Method
$\left(2x^2+3x-1\right)\left(6x^2-7x-5\right)$ $2x^2\left(6x^2-7x-5\right)+3x\left(6x^2-7x-5\right)-1\left(6x^2-7x-5\right)$ $\left(12x^4-14x^3-10x^2\right)+\left(18x^3-21x^2-15x\right)+\left(-6x^2+7x+5\right)$ $12x^4+4x^3-37x^2-8x+5$	(box table below) **combine terms on the diagonals of the unshaded boxes(top right to lower left)* $12x^4+4x^3-37x^2-8x+5$

	$6x^2$	$-7x$	-5
$2x^2$	$12x^4$	$-14x^3$	$-10x^2$
$3x$	$18x^3$	$-21x^2$	$-15x$
-1	$-6x^2$	$7x$	5

Vertical Method

$$
\begin{array}{rrrrr}
 & & 2x^2 & +3x & -1 \\
\times & & 6x^2 & -7x & -5 \\
\hline
 & & 10x^2 & +15x & +5 \\
 & -14x^3 & -21x^2 & -15x & \\
12x^4 & +18x^3 & -6x^2 & & \\
\hline
12x^4 & +4x^3 & -37x^2 & -8x & +5
\end{array}
$$

Practice Exercises B

Perform the required operations. Write your answers in standard form and determine if the result is a polynomial.

1. $(6x-3)(-5x-6)$
2. $\left(3x-5\right)\left(x+2\right)$
3. $(7x+2)(10x+5)$
4. $\left(2x+3\right)\left(4x+1\right)$
5. $(-4x-5)(9x+8)$
6. $\left(3x-y\right)\left(3x+y\right)$
7. $\left(2x+7\right)^2$
8. $\left(3-5x\right)^2$
9. $\left(5x^3-1\right)^2$
10. $\left(2x^3-3y\right)\left(2x^3+3y\right)$
11. $\left(x^2-2x+3\right)\left(x+4\right)$
12. $\left(x^2+3x-2\right)\left(x-3\right)$

13. $(2x+7)(5x^2+4x+1)$

14. $\left(-3x^2-5\right)\left(x^2+7x+12\right)$

15. $(-9x-2)(-3x^2-8x-5)$

16. $(2x^2+4x+10)(3x-4)$

17. $\left(x^2+x-3\right)\left(x^2+x+1\right)$

18. $\left(2x^2-3x+1\right)\left(x^2+x+1\right)$

19. $(x^2-8x-1)(2x^2+10x+4)$

20. $\left(x^2-3x+7\right)\left(3x^2+5x-3\right)$

21. $\left(y^2+2y-3\right)\left(5y^2+3y+4\right)$

22. $(-3x^2+x+3)(2x^2+10x+4)$

23. $(4x^2+6x+1)(-5x^2-3x-6)$

24. $\left(4x^3-x^2+3x\right)\left(x^3+12x-3\right)$

YOU DECIDE

Are polynomials closed under addition, subtraction, multiplication? Justify your conclusion using the method of your choice.

Unit 2 Cluster 5: Polynomials
(A.APR.2, A.APR.3, F.IF.7c, and N.CN.9)

Cluster 5: Relationships between zeros and factors of polynomials

2.5 Know and apply the Remainder Theorem

2.5 Identify the zeros of polynomials and use the zeros to construct a rough graph.

Cluster 10: Analyze functions using different representations

2.10 Graph polynomial functions identifying zeros and showing end behavior

Cluster 1: Use Complex Numbers in Polynomial Identities and Equations

2.1 Know the Fundamental Theorem of Algebra

Remainder Theorem
For a polynomial $p(x)$ and a number *a*, the remainder when dividing by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $x-a$ is a factor of $p(x)$.

Example 1:

Is $x+5$ a factor of $f(x)=3x^2+14x-5$?

$x-a$ $x+5=x-(-5)$ $a=-5$	Identify *a*.
$f(x)=3x^2+14x-5$ $f(-5)=3(-5)^2+14(-5)-5$ $=75-70-5$ $=0$	Substitute *a* in for *x*. Simplify
Since $f(-5)=0$ the binomial $x+5$ is a factor of $f(x)=3x^2+14x-5$.	
NOTE: If we factored $f(x)=3x^2+14x-5$, the result would be $f(x)=(3x-1)(x+5)$.	

Example 2:

Is $x-3$ a factor of $f(x)=2x^2-7x-4$?

$x-a$ $x-3$ $a=3$	Identify *a*.
$f(x)=2x^2-7x-4$ $f(3)=2(3)^2-7(3)-4$ $=18-21-4$ $=-7$	Substitute *a* in for *x*. Simplify
Since $f(3)\neq 0$ the binomial $x-3$ is not a factor of $f(x)=2x^2-7x-4$.	
NOTE: If we factored $f(x)=2x^2-7x-4$, the result would be $f(x)=(2x+1)(x-4)$.	

Example 3:

Is $x+2$ a factor of $f(x)=x^3-3x^2-6x+8$?

$x-a$ $x+2=x-(-2)$ $a=-2$	Identify *a*.
$f(x)=x^3-3x^2-6x+8$ $f(-2)=(-2)^3-3(-2)^2-6(-2)+8$ $=-8-12+12+8$ $=0$	Substitute *a* in for *x*. Simplify
Since $f(-2)=0$ the binomial $x+2$ is a factor of $f(x)=x^3-3x^2-6x+8$.	
NOTE: If we factored $f(x)=x^3-3x^2-6x+8$, the result would be $f(x)=(x+2)(x-1)(x-4)$.	

Practice Exercises A

For the given polynomials determine which of the binomials listed are factors.

1. $f(x)=-2x^2+15x-7$
   a. $x+1$
   b. $x-7$
   c. $x-2$

2. $f(x)=3x^2-7x-6$
   a. $x-3$
   b. $x+2$
   c. $x+1$

3. $f(x)=x^3+3x^2-4x-12$
   a. $x+2$
   b. $x-2$
   c. $x+3$

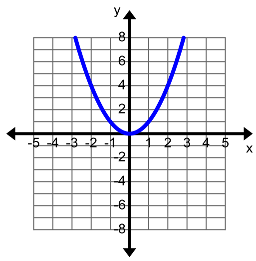
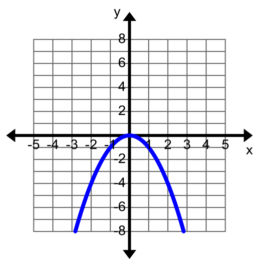
4. $f(x)=2x^3+15x^2+22x-15$
   a. $x+3$
   b. $x+5$
   c. $x-3$

Graphing Polynomials

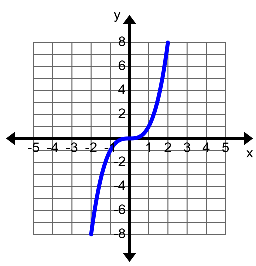
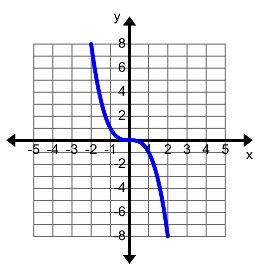
Using a graphing calculator it is easy to see the end behavior, identify the zeros, and the basic shape of any polynomial.

Power Function Graphs and End Behavior

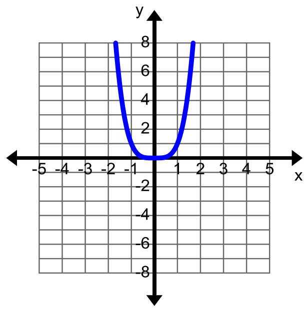
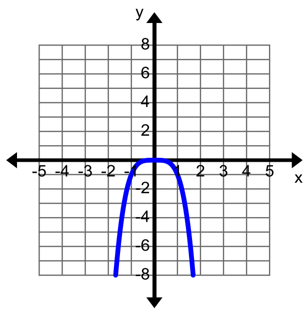
Quadratic (2nd Degree)

Function	Graph	Left End Behavior	Right End Behavior
$f(x)=x^2$		$\lim_{x\to-\infty} f(x)=+\infty$	$\lim_{x\to\infty} f(x)=+\infty$
$f(x)=-x^2$		$\lim_{x\to-\infty} f(x)=-\infty$	$\lim_{x\to\infty} f(x)=-\infty$

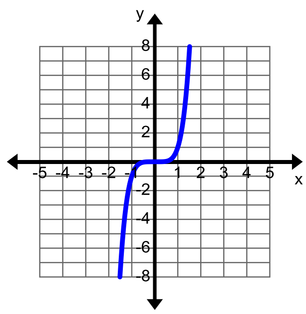
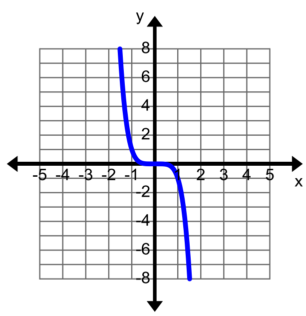
Cubic (3rd Degree)

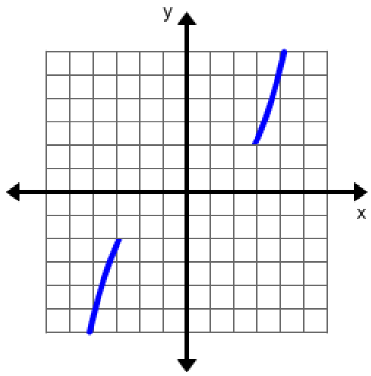
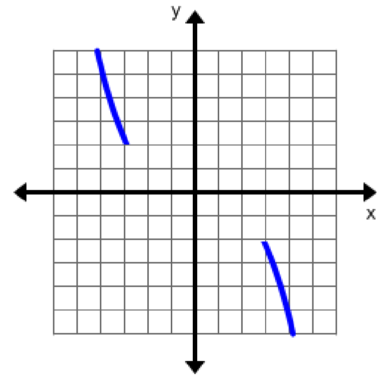
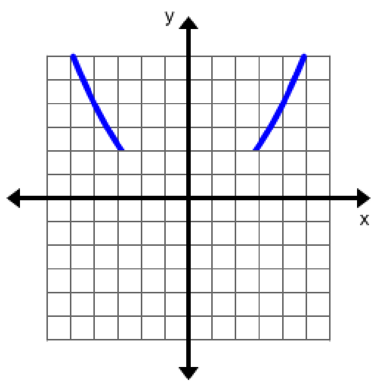
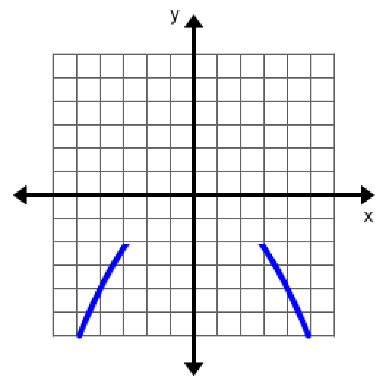
Function	Graph	Left End Behavior	Right End Behavior
$f(x)=x^3$		$\lim_{x\to-\infty} f(x)=-\infty$	$\lim_{x\to\infty} f(x)=+\infty$
$f(x)=-x^3$		$\lim_{x\to-\infty} f(x)=+\infty$	$\lim_{x\to\infty} f(x)=-\infty$

Quartic (4^{th} Degree)

Function	Graph	Left End Behavior	Right End Behavior
$f(x) = x^4$		$\lim\limits_{x \to -\infty} f(x) = +\infty$	$\lim\limits_{x \to \infty} f(x) = +\infty$
$f(x) = -x^4$		$\lim\limits_{x \to -\infty} f(x) = -\infty$	$\lim\limits_{x \to \infty} f(x) = -\infty$

Quintic (5^{th} Degree)

Function	Graph	Left End Behavior	Right End Behavior
$f(x) = x^5$		$\lim\limits_{x \to -\infty} f(x) = -\infty$	$\lim\limits_{x \to \infty} f(x) = +\infty$
$f(x) = -x^5$		$\lim\limits_{x \to -\infty} f(x) = +\infty$	$\lim\limits_{x \to \infty} f(x) = -\infty$

Generalizing Polynomial End Behavior $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$	
***n* is odd**	
$a_n>0$  $\lim\limits_{x\to-\infty} f(x)=-\infty$ $\lim\limits_{x\to\infty} f(x)=+\infty$	$a_n<0$  $\lim\limits_{x\to-\infty} f(x)=+\infty$ $\lim\limits_{x\to\infty} f(x)=-\infty$
***n* is even**	
$a_n>0$  $\lim\limits_{x\to-\infty} f(x)=+\infty$ $\lim\limits_{x\to\infty} f(x)=+\infty$	$a_n<0$  $\lim\limits_{x\to-\infty} f(x)=-\infty$ $\lim\limits_{x\to\infty} f(x)=-\infty$

Example 4: Without graphing, determine the end behavior of each polynomial.

a. $f(x)=-x^3+6x^2-5x+7$

b. $f(x)=2x^4+6x^2+7$

a. $f(x)=-x^3+6x^2-5x+7$ Left end behavior: $\lim\limits_{x\to-\infty} f(x)=+\infty$ Right end behavior: $\lim\limits_{x\to\infty} f(x)=-\infty$	$n=3$ (odd) and $a_n=-1<0$

b. $f(x)=2x^4+6x^2+7$ Left end behavior: $\lim\limits_{x\to-\infty} f(x)=+\infty$ Right end behavior: $\lim\limits_{x\to\infty} f(x)=+\infty$	$n=4$ (even) and $a_n=2>0$

Practice Exercises B

Without graphing, determine the end behavior of each polynomial.

1. $f(x) = 2x^5 + 7x^3 - 4x$
2. $f(x) = -3x^6 - 8x^5 + 2x$
3. $f(x) = 4x^7 + 5$
4. $f(x) = -10x^3 - 3x^2 - 5$
5. $f(x) = -6x^{10} + 5x^4 - 5x^3 + 9$
6. $f(x) = 8x^4 + 10x^3 + 3x - 4$

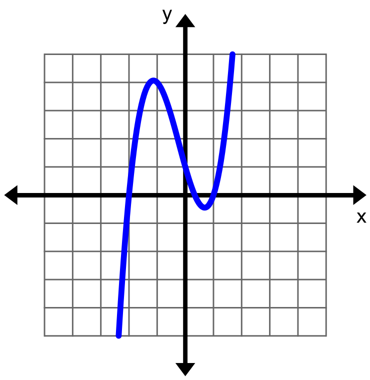
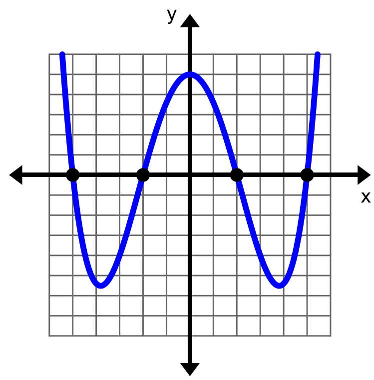
Fundamental Theorem of Algebra

A polynomial function of degree $n > 0$ has n complex zeros (every real number is a complex number i.e., $3 = 3 + 0i$). Some of the zeros may be repeated.

Example 5: Determine the number of zeros each of the following polynomials have.

a. $f(x) = 3x^3 + 2x^2 - 7x + 2$

b. $g(x) = 100 + x^4 - 29x^2$

a.	b.
$f(x)$ is a 3rd degree polynomial function, therefore, the function will have three zeros.  You can see from the graph that the function crosses the *x*-axis three times.	$g(x)$ is a 4th degree polynomial function, therefore, the function will have four zeros.  You can see from the graph that the function crosses the *x*-axis four times.

Practice Exercises C

Without graphing, determine the number of zeros for each of the following polynomials.

1. $f(x) = 2x^4 - 5x^3 - 26x^2 - x + 30$
2. $f(x) = 3x^3 + x^2 - 62x + 40$
3. $f(x) = -4x^5 + 52x^3 - 144x$
4. $f(x) = 2x^4 + 5x^3 - 35x^2 - 80x + 48$
5. $f(x) = -x^6 + 14x^4 - 49x^2 + 36$
6. $f(x) = 5x^7 - 70x^5 + 245x^3 - 180x + 5$

Example 6: Use technology to graph the polynomial. Identify its zeros and end behavior.

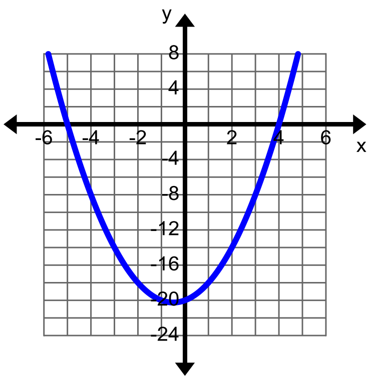
a. $f(x) = x^2 - x - 20$
$f(x) = (x-4)(x+5)$

b. $f(x) = -x^3 + 2x^2 + 5x - 6$
$f(x) = -(x-3)(x+2)(x-1)$

c. $f(x) = -x^4 + x^3 + 11x^2 - 9x - 18$
$f(x) = -(x^2 - 9)(x+1)(x-2)$

d. $f(x) = x^5 + 3x^4 - 20x^3 - 60x^2 + 64x + 192$
$f(x) = (x^2 - 4)(x+3)(x-4)(x+4)$

a. $f(x) = x^2 - x - 20$
$f(x) = (x-4)(x+5)$

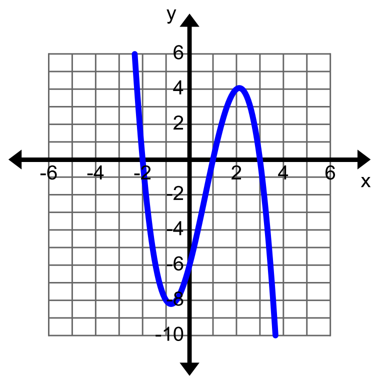


The zeros are: $(-5, 0)$ and $(4, 0)$

Left end behavior: $\lim_{x \to -\infty} f(x) = +\infty$

Right end behavior: $\lim_{x \to \infty} f(x) = +\infty$

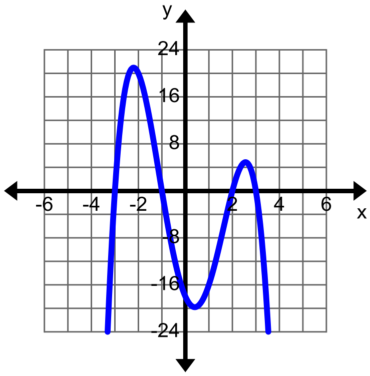
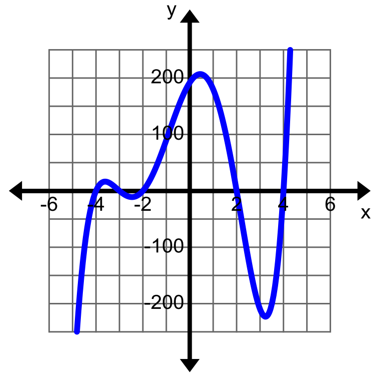
b. $f(x) = -x^3 + 2x^2 + 5x - 6$
$f(x) = -(x-3)(x+2)(x-1)$



The zeros are: $(-2, 0)$, $(1, 0)$, and $(3, 0)$

Left end behavior: $\lim_{x \to -\infty} f(x) = +\infty$

Right end behavior: $\lim_{x \to \infty} f(x) = -\infty$

c. $f(x) = -x^4 + x^3 + 11x^2 - 9x - 18$ $f(x) = -(x^2 - 9)(x+1)(x-2)$  The zeros are: $(-3,0)$, $(-1,0)$, $(2,0)$, $(3,0)$ Left end behavior: $\lim\limits_{x\to-\infty} f(x) = -\infty$ Right end behavior: $\lim\limits_{x\to\infty} f(x) = -\infty$	d. $f(x) = x^5 + 3x^4 - 20x^3 - 60x^2 + 64x + 192$ $f(x) = (x^2 - 4)(x+3)(x-4)(x+4)$  The zeros are: $(-4,0)$, $(-3,0)$, $(-2,0)$, $(2,0)$, $(4,0)$ Left end behavior: $\lim\limits_{x\to-\infty} f(x) = -\infty$ Right end behavior: $\lim\limits_{x\to\infty} f(x) = +\infty$

Practice Exercises D

Use technology to graph the polynomial. Identify its zeros and end behavior.

1. $f(x) = (x+9)(x-10)(x-6)(x+2)$
   $f(x) = x^4 - 5x^3 - 98x^2 + 372x + 1080$

2. $f(x) = (x^2 - 9)(x+4)$
   $f(x) = x^3 + 4x^2 - 9x - 36$

3. $f(x) = (-3x)(x^2 - 1)(x^2 - 25)$
   $f(x) = -3x^5 + 78x^3 - 75x$

4. $f(x) = (x^2 - 1)(x^2 - 4)(x^2 - 25)$
   $f(x) = -x^6 + 30x^4 - 129x^2 + 100$

5. $f(x) = -2(x+3)(x-7)(x+6)$
   $f(x) = -2x^3 - 4x^2 + 90x + 252$

6. $f(x) = (5x)(x^2 - 1)(x^2 - 9)$
   $f(x) = 5x^5 - 50x^3 + 45x$

7. $f(x) = (2x+3)(x-1)(x-5)$
   $f(x) = 2x^3 - 8x^2 - 8x + 15$

8. $f(x) = (3x+2)(x-6)(x^2 - 4)$
   $f(x) = 3x^4 - 16x^3 - 24x^2 + 64x + 48$

Multiplicity of Zeros

Exploration:

Given: $f(x)=(x-3)(x+2)^2$

a. Identify the zeros of $f(x)$.

b. According to the Fundamental Theorem of Algebra how many zeros should $f(x)$ have?

c. Is there a difference between the number of zeros found in part a and the expected number of zeros found in part b?

a. $(x-3)(x+2)^2=0$	Set the function equal to zero.
$(x-3)(x+2)(x+2)=0$	Remember that $(x+2)^2=(x+2)(x+2)$.
$x-3=0$ $\quad x+2=0$ $\quad x+2=0$ $x=3$ $\quad x=-2$ $\quad x=-2$ $(3,0)$ $\quad (-2,0)$ $\quad (-2,0)$	Set each factor equal to zero. Solve for *x*. Write the zeros as ordered pairs.
The zeros of $f(x)$ are $(3,0)$ and $(-2,0)$ because $(-2,0)$ repeats.	

b. The expanded form of $f(x)$ is $f(x)=x^3+x^2-8x-12$. $f(x)$ is a 3rd degree polynomial function, therefore it should have three zeros.

c. No, three zeros were found, but only two were listed because one of the zeros repeats.

Multiplicity of Zeros: Recall that the Fundamental Theorem of Algebra states that zeros can be repeated. When a zero is repeated, the same factor occurs multiple times. We say the factor has a multiplicity of the number of times it is repeated. For instance, $(x-5)^4$ means that the zero $(5,0)$ is repeated four times and has a multiplicity of 4.

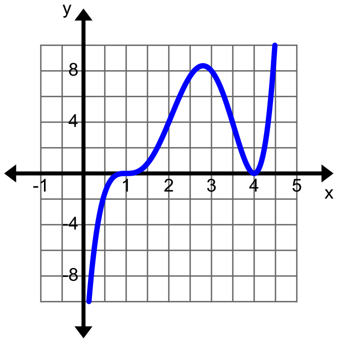
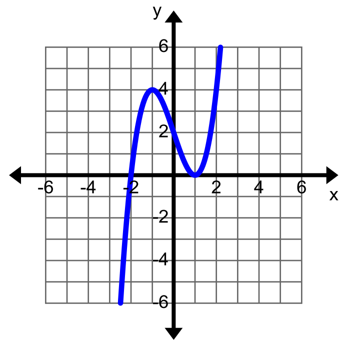
Example 7:

Use technology to graph the polynomial. Identify the zeros, their multiplicity, and determine whether they touch or cross the *x*-axis at each zero.

a. $f(x)=(x-4)^2(x-1)^3$

b. $f(x)=(x-1)^2(x+2)$

a.

 The zeros are $(1,0)$ and $(4,0)$. The zero $(1,0)$ has multiplicity three and it crosses the *x*-axis. The zero $(4,0)$ has multiplicity two and it touches the *x*-axis.	 The zeros are $(1,0)$ and $(-2,0)$. The zero $(1,0)$ has multiplicity two and it touches the *x*-axis. The zero $(-2,0)$ has multiplicity of one and it crosses the *x*-axis.

NOTE: When the multiplicity of a zero is even, the graph of the function touches the *x*-axis. When the multiplicity of a zero is odd, the graph of the function crosses the *x*-axis.

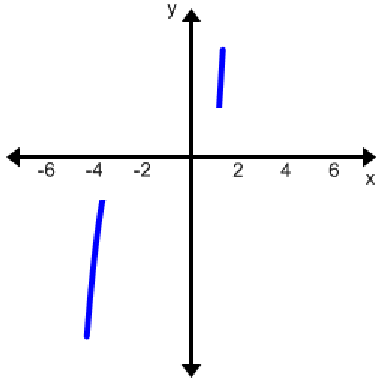
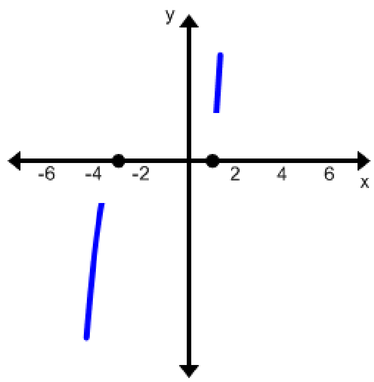
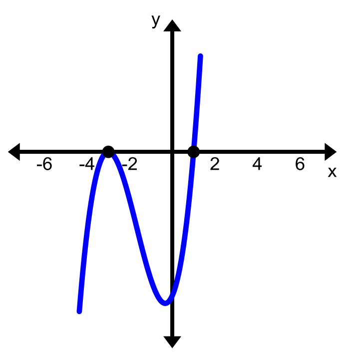
Practice Exercises E

Use technology to graph the polynomial. Identify the zeros, their multiplicity, and determine whether they touch or cross the *x*-axis at each zero.

1. $f(x)=(x+1)^4(x-5)^3$
2. $f(x)=(x-3)(x+2)^2$
3. $f(x)=(x^2-4)(x+5)^3(x-1)^2$
4. $f(x)=(x-1)^3(x^2-9)$
5. $f(x)=(x-2)^2(x+3)^2(x-4)$
6. $f(x)=(x-1)^2(x-4)^3$
7. $f(x)=(x-4)^2(x+1)^3(x+3)$
8. $f(x)=(x-2)^3(x+1)^2(x-5)^2$

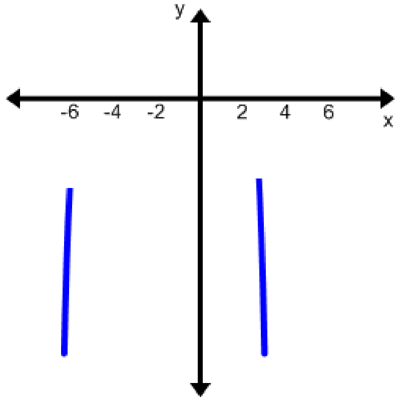
Example 8:
Without using technology, sketch each polynomial.

a. $f(x)=(x+3)^2(x-1)$ b. $f(x)=-x(x+5)^4(x-2)^3$

<table>
<tr><td>a. $f(x)=(x+3)^2(x-1)$</td><td></td></tr>
<tr><td>End Behavior

</td><td>$f(x)$ is a 3rd degree polynomial.
$a_n>0$ (Leading coefficient is positive)</td></tr>
<tr><td>
</td><td>Zeros: $(-3,0)$ and $(1,0)$</td></tr>
<tr><td>
</td><td>
<table>
<tr><th>Zero</th><th>Multiplicity</th><th>Touch /Cross</th></tr>
<tr><td>$(-3,0)$</td><td>2</td><td>Touches</td></tr>
<tr><td>$(1,0)$</td><td>1</td><td>Crosses</td></tr>
</table>
</td></tr>
</table>

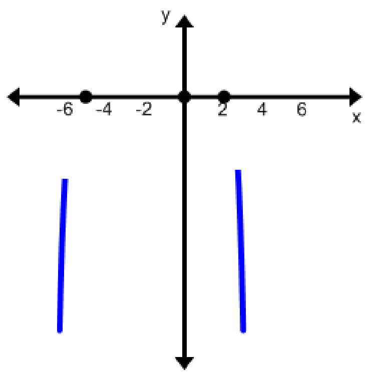
b. $f(x) = -x(x+5)^4(x-2)^3$

End Behavior

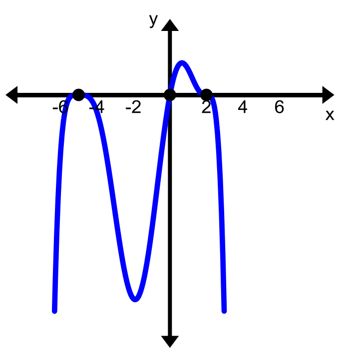


$f(x)$ is an 8^{th} degree polynomial.

$a_n < 0$ (Leading coefficient is negative)



Zeros: $(-5,0),(0,0)$, and $(2,0)$



Zero	Multiplicity	Touch /Cross
$(-5,0)$	4	Touches
$(0,0)$	1	Crosses
$(2,0)$	3	Crosses

Practice Exercises F

Without using technology, sketch each polynomial.

1. $f(x) = x^3 - 4x$
2. $f(x) = (x^2 - 4)(x^2 - 1)$
3. $f(x) = (x^2 - 1)(x^2 - 9)(x + 2)$
4. $f(x) = (x^2 - 1)(x^2 - 9)(x^2 - 4)$
5. $f(x) = (x - 4)^2 (x + 2)$
6. $f(x) = (x - 3)^2 (x + 5)^2 (x - 1)$
7. $f(x) = (x + 2)^4 (x - 1)^5$
8. $f(x) = (x - 4)^3 (x + 1)^2$

Unit 2 Cluster 6: Polynomials
(A.APR.4, A.APR.5, and N.CN.8)

Cluster 6: Polynomial identities
- 6.1 Prove polynomial identities
- 6.2 Know and apply the binomial theorem

Cluster 1: Use complex numbers in polynomial identities and equations
- 1.1 Extend polynomial identities to the complex numbers

Polynomial Identities	
Perfect Square Trinomial $(A+B)^2 = A^2+2AB+B^2$	$(4x+3y)^2 = (4x)^2+2(4x)(3y)+(3y)^2$ $= 16x^2+24xy+9y^2$
Difference of Squares $(A+B)(A-B)=A^2-B^2$	$(2x+5y)(2x-5y)=(2x)^2-(5y)^2$ $=4x^2-25y^2$
Cubic Polynomials $(A+B)^3 = A^3+3A^2B+3AB^2+B^3$ $(A-B)^3 = A^3-3A^2B+3AB^2-B^3$	$(2x+5y)^3=(2x)^3+(3)(2x)^2(5y)+(3)(2x)(5y)^2+(5y)^3$ $=8x^3+60x^2y+150xy^2+125y^3$ $(2x-5y)^3=(2x)^3-(3)(2x)^2(5y)+(3)(2x)(5y)^2-(5y)^3$ $=8x^3-60x^2y+150xy^2-125y^3$
Sum and Difference of Cubes $A^3+B^3=(A+B)(A^2-AB+B^2)$ $A^3-B^3=(A-B)(A^2+AB+B^2)$	$27x^3+64y^3=(3x+4y)\left[(3x)^2-(3x)(4y)+(4y)^2\right]$ $=(3x+4y)(9x^2-12xy+16y^2)$ $27x^3-64y^3=(3x-4y)\left[(3x)^2+(3x)(4y)+(4y)^2\right]$ $=(3x-4y)(9x^2+12xy+16y^2)$
Trinomial Leading Coefficient 1 $x^2+(a+b)x+ab=(x+a)(x+b)$	$x^2+5x+6=x^2+(2+3)x+(2)(3)$ $=(x+2)(x+3)$ $x^2-5x+6=x^2+(-2-3)x+(-2)(-3)$ $=(x-2)(x-3)$

Quadratic Formula Given $ax^2+bx+c=0$ $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$	$2x^2-4x-5=0$ $x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(2)(-5)}}{2(2)}$ $=\dfrac{4\pm\sqrt{16+40}}{4}$ $=\dfrac{4\pm\sqrt{56}}{4}$ $=\dfrac{4\pm2\sqrt{14}}{4}$ $=\dfrac{2\pm\sqrt{14}}{2}$	$4x^2+9$ $x=\dfrac{-0\pm\sqrt{0^2-4(4)(9)}}{2(4)}$ $=\dfrac{\pm\sqrt{-144}}{8}$ $=\dfrac{\pm12i}{8}$ $=\dfrac{\pm3i}{2}$
Sum of Squares $A^2+B^2=(A+Bi)(A-Bi)$	$4x^2+9=(2x+3i)(2x-3i)$	

VOCABULARY

A polynomial with integer coefficients that cannot be factored into polynomials of lower degree, also with integer coefficients, is called an **irreducible or prime polynomial**.

Practice Exercises A

Multiply the expressions using the polynomial identities.

1. $(x-5)(x+4)$
2. $(2x-12y)^2$
3. $(4x-y)^3$
4. $(2x+3)(4x^2-6x+9)$
5. $(2x-5)^3$
6. $(8x-1)^2$
7. $(9x-8y)(9x+8y)$
8. $(4x-5)(16x^2+20x+25)$
9. $(x-13)(x-3)$
10. $(x+3)^3$
11. $(13x+8i)(13x-8i)$
12. $(12x-13y)(12x+13y)$
13. $(10x+4i)(10x-4i)$
14. $(x+12)(x+2)$
15. $(x-11y)(x^2+11y+121y^2)$
16. $(6x+7)^2$
17. $(4x-11)(4x+11)$
18. $(9x+i)(9x-i)$

Practice Exercises B

Factor the expressions using the polynomial identities.

1. $27x^3 - y^3$	2. $81x^2 - 18xy + y^2$	3. $27x^3 - 54x^2y + 36xy^2 - 8y^3$
4. $4x^2 - 49$	5. $9x^2 + 64$	6. $4x^2 + 52x + 169$
7. $x^2 + 19x + 88$	8. $16x^2 - 100y^2$	9. $343x^3 + 8y^3$
10. $36x^2 + 60x + 25$	11. $x^3 - 15x^2 + 75x - 125$	12. $25x^2 - 121$
13. $144x^2 + 25$	14. $x^3 - 512$	15. $x^2 + 4x - 45$
16. $x^3 + 3x^2y + 3xy^2 + y^3$	17. $x^2 - 9x + 18$	18. $16x^2 + 49$

Use the quadratic formula to solve each equation.

19. $x^2 - 5x - 3 = 0$	20. $-4x^2 + 3x + 1 = 0$	21. $x^2 - x - 2 = 0$
22. $-5x^2 - 2x + 3 = 0$	23. $3x^2 + 7x + 2 = 0$	24. $x^2 + 10x + 11 = 0$

Example 1: Factor $x^2 - 6x + 10$ over the complex numbers.

$x^2 - 6x + 10$	This is an irreducible polynomial. We will have to use the quadratic formula to find the roots for this polynomial.
$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$ $= \dfrac{6 \pm \sqrt{36 - 40}}{2}$ $= \dfrac{6 \pm \sqrt{-4}}{2}$ $= \dfrac{6 \pm 2i}{2}$ $= 3 \pm i$	$a = 1$ $b = -6$ $c = 10$
$[x - (3+i)][x - (3-i)]$	Write as factors $(x - di)(x - fi)$
$(x - 3 - i)(x - 3 + i)$	Simplify

Example 2: Factor x^3-8 over the complex numbers.

x^3-8	This is a difference of cubes. $A^3-B^3=(A-B)(A^2+AB+B^2)$ $A=\sqrt[3]{x^3}=x$ $B=\sqrt[3]{8}=2$
$x^3-8=(x-2)(x^2+2x+4)$	Substitute the values for *A* and *B* into the polynomial identity and simplify.
x^2+2x+4	This is an irreducible polynomial. We will have to use the quadratic formula to find the roots for this polynomial.
$x=\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}$ $=\frac{-2\pm\sqrt{4-16}}{2}$ $=\frac{-2\pm\sqrt{-12}}{2}$ $=\frac{-2\pm 2i\sqrt{3}}{2}$ $=-1\pm i\sqrt{3}$	$a=1$ $b=2$ $c=4$
$\left[x-\left(-1+i\sqrt{3}\right)\right]\left[x-\left(-1-i\sqrt{3}\right)\right]$	Write as factors $(x-di)(x-fi)$
$\left(x+1-i\sqrt{3}\right)\left(x+1+i\sqrt{3}\right)$	Simplify
$x^3-8=(x-2)(x^2+2x+4)$ $=(x-2)\left(x+1-i\sqrt{3}\right)\left(x+1+i\sqrt{3}\right)$	Write all of the factors as one expression.

Practice Exercises C

Factor each expression over the complex numbers.

1. x^2-4x+5
2. $x^2-2x+10$
3. x^2+4x+8
4. $x^2+8x+17$
5. x^2+4x+7
6. x^2+5
7. x^2+6
8. x^2+8
9. x^3+27
10. x^3-64
11. x^3+1
12. $(4x^2-25)(x^2+9)$
13. $(x^2+5x+6)(x^2+4)$
14. $(4x^2-12x+9)(x^2+3)$
15. $(x^2-2x+3)(x^2+1)$

Binomial Theorem

Exploration 1:

1. Expand $(x+1)^0$

2. Expand $(x+1)^1$

3. Expand $(x+1)^2$

4. Expand $(x+1)^3$

5. Expand $(x+1)^4$

6. Write the coefficients of each expansion.

7. Without expanding, determine what the coefficients of the expansion of $(x+1)^5$.

Answer:

1. $(x+1)^0 = 1$
2. $(x+1)^1 = x+1$
3. $(x+1)^2 = x^2 + 2x + 1$
4. $(x+1)^3 = x^3 + 3x^2 + 3x + 1$
5. $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$
6. 1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
7. 1 5 10 10 5 1

Pascal's Triangle was named in honor of Blaise Pascal. The pattern in the triangle represents the coefficients for a binomial expansion.

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1
Row 4	1 4 6 4 1
Row 5	1 5 10 10 5 1

Exploration 2
Complete the pattern for Row 6, Row 7, and Row 8.

The Binomial Theorem states that for any positive integer *n*,
$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n$, where $\binom{n}{r} = {}_nC_r$. Although the binomial theorem uses combinations to determine the coefficients for each term, Pascal's triangle can be used to determine the coefficients instead.

Notice the *a* exponents decrease from *n* to 0, while the *b* exponents increase from 0 to *n*.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a^1b^0 + a^0b^1$$
$$(a+b)^2 = a^2b^0 + 2a^1b^1 + a^0b^1$$
$$(a+b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$
$$(a+b)^4 = a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + a^0b^4$$
$$(a+b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

Example 3:
Expand $(3x-2y)^5$

$(3x-2y)^5$: *Use* $(a+b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$ *where* $a = 3x$ and $b = -2y$
$(3x)^5(-2y)^0 + 5(3x)^4(-2y)^1 + 10(3x)^3(-2y)^2 + 10(3x)^2(-2y)^3 + 5(3x)^1(-2y)^4 + (3x)^0(-2y)^5$ $243x^5 + 5(81x^4)(-2y) + 10(27x^3)(4y^2) + 10(9x^2)(-8y^3) + 5(3x)(16y^4) + (-32y^5)$ $243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$

Practice Exercises D

Expand each of the binomials using the Binomial Theorem.

1. $(x+3)^6$
2. $(x-2)^7$
3. $(2x-1)^5$
4. $(5x+y)^6$
5. $(4x-3y)^5$
6. $(3x+2y)^4$

Series

Unit 2 Cluster 3 (A.SSE.4): Geometric Series

Cluster 3: Write expressions in equivalent forms to solve problems.

2.3 Derive the formula for the sum of a geometric series (when the common ratio is not 1) and use the formula to solve problems.

H.2.3 Discover and justify the formula for infinite geometric series.

H.2.3 Discover and justify the formula for a finite arithmetic series.

VOCABULARY

A **sequence**, $\{a_1, a_2, a_3, a_4, \ldots a_n, \ldots\}$, is a list of terms in a specified order where a_1 is the first term and a_n is the ***n*th term** or the **general term**. A **finite sequence** has a first term and a last term while an **infinite sequence** has a first term, but continues without end.

A **series**, $S = a_1 + a_2 + a_3 + a_4 + \ldots a_n + \ldots$, is the sum of the terms of a sequence. A **finite series** sums the terms of a finite sequence while an **infinite series** sums the terms of an infinite sequence.

A **geometric** series is the sum of the terms of a geometric sequence, $S = a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots$, where a is the first term and r is the common ratio.

The Greek capital letter sigma, $\sum$, can be used to indicate the sum of a sequence. The sigma is followed by the explicit formula for the sequence. The small number below the sigma is the first term (lower limit) and the small number above the sigma is the last term (upper limit). The numbers from the lower limit to the upper limit are substituted into the explicit formula and added together. For example,

$$\sum_{k=3}^{7} k^3 = 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = 784.$$

For an infinite series the upper limit is ∞.

Upper Limit

$$\sum_{k=1}^{n} 2k+1 \longleftarrow \text{Explicit Formula}$$

Lower Limit

Sigma Notation

Example 1:

Write out and evaluate each sum.

a. $\displaystyle\sum_{k=1}^{5} k^2$ b. $\displaystyle\sum_{k=3}^{8} 2k$ c. $\displaystyle\sum_{k=1}^{4} \frac{k}{k+1}$

a. $\displaystyle\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$ $= 1 + 4 + 9 + 16 + 25$ $= 55$	Substitute values into k beginning with 1 and ending with 5. Sum each term.

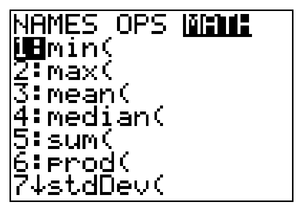
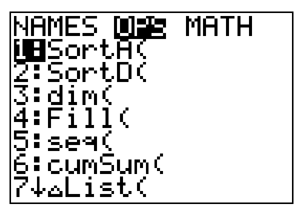
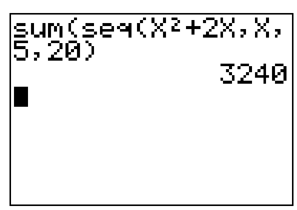
b. $\sum_{k=3}^{8} 2k = (2\cdot 3)+(2\cdot 4)+(2\cdot 5)+(2\cdot 6)+(2\cdot 7)+(2\cdot 8)$ $= 6+8+10+12+14+16$ $= 66$	Substitute values into k beginning with 3 and ending with 8. Sum each term.

c. $\sum_{k=1}^{4} \frac{k}{k+1} = \frac{1}{1+1}+\frac{2}{2+1}+\frac{3}{3+1}+\frac{4}{4+1}$ $= \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$ $= \frac{163}{60}$	Substitute values into k beginning with 1 and ending with 4. Sum each term.

Example 2:

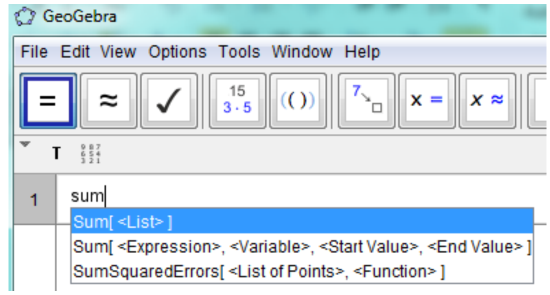
Use technology to find the sum of $\sum_{k=5}^{20} k^2 + 2k$.

TI-83 or TI-84 Graphing Calculator

Push 2ND LIST STAT then use your arrow keys to arrow over to the MATH menu. Option number 5 is sum(. This will sum all of the terms of the sequence k^2+2k. Select it by pushing L5 5 U or use your arrow keys to arrow down to 5 and then push ENTRY SOLVE ENTER.	
Push 2ND LIST STAT again then use your arrow keys to arrow over to the OPS menu. Option number 5 is seq(. This will allow you to enter the sequence so that its terms can be summed. Select it by pushing L5 5 U or use your arrow keys to arrow down to 5 and then push ENTRY SOLVE ENTER.	
To enter the sequences use the syntax seq(explicit formula, variable, lower limit, upper limit) you can use the variable x to enter sum(seq(x^2+2x, x, 5, 20). Once you have all of the information entered, push ENTRY SOLVE ENTER and the calculator will find the sum.	

Use technology to find the sum of $\sum_{k=5}^{20} k^2 + 2k$.

Geogebra CAS & Graphics

<table>
<tr>
<td>To bring up the CAS & Graphics window push Ctrl+Shift+k. Begin typing the word sum and options will appear. Select Sum[<Expression>, <Variable>, <Start Value>, <End Value>].</td>
<td>
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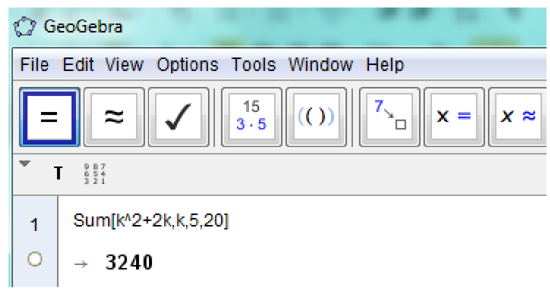
<tr>
<td>The expression is $k^2 + 2k$, the variable is k, the start value is 5, and the end value is 20 (push tab after entering each part). Once all the information is in, push enter and the sum will appear.</td>
<td>
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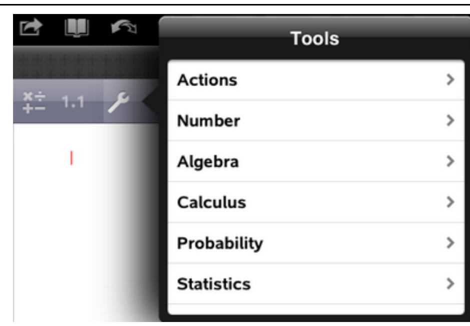
Use technology to find the sum of $\sum_{k=5}^{20} k^2 + 2k$.

TI-*N*spire CAS on the iPad

<table>
<tr>
<td>Select a new document by pushing the + at the top left corner. A drop down menu will appear then select Calculator. Bring up the tools menu by pushing the wrench. The tools we are using are under Statistics so select Statistics.</td>
<td>
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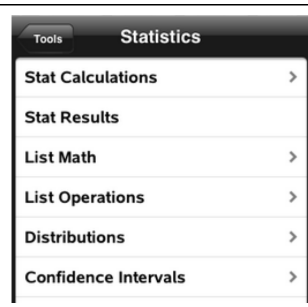
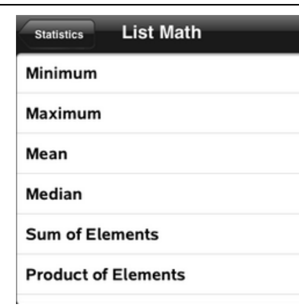
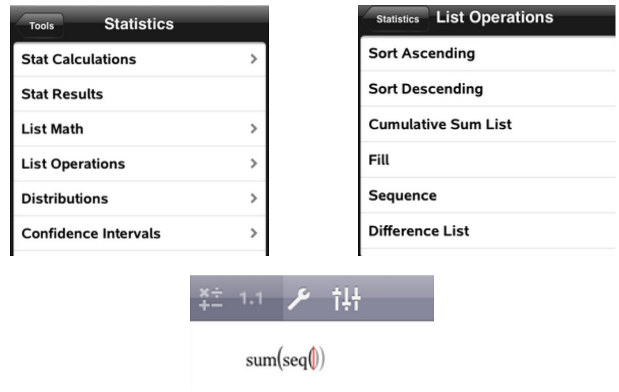
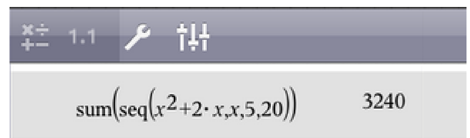
<tr>
<td>The sum option is under the List Math menu. Select it then select the Sum of Elements. This will sum all the terms of a sequence.</td>
<td>



sum()</td>
</tr>
</table>

Bring the tools menu up again by pushing the wrench. Push on Statistics at the top left corner so that the menu goes back to the Statistics menu. Select List Operations. We need to enter a sequence so select Sequence.	
Use the syntax of seq(explicit formula, variable, lower limit, upper limit). Push enter and the sum will appear.	

Example 3:
Write each series using sigma notation.

a. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}...+\frac{1}{10}$ b. $12+18+24+...+54$ c. $50+48+46+...+30$

a. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}...+\frac{1}{10}$	Determine the explicit formula. In this case it is $\frac{1}{k}$.
$\sum_{k=3}^{10}\frac{1}{k}$	The first term is $\frac{1}{3}$ so the lower limit is 3. The last term is $\frac{1}{10}$ so the upper limit is 10.

b. $12+18+24+...+72$	Determine the explicit formula. This is an arithmetic series. The common difference, *d*, is 6 and the first term is 12. The explicit formula is $6k+6$ or $6k$, depending on the lower limit.
$\sum_{k=2}^{12}6k$ or $\sum_{k=1}^{11}6k+6$	The first term is 12 which is $6(1)+6$ or $6(2)$ so the lower limit can be 1 or 2. The last term is 72 which is $6(11)+6$ or $6(12)$ so the upper limit can be 11 or 12.

c. $50+48+46+...+30$	Determine the explicit formula. This is an arithmetic series. The common difference, *d*, is -2 and the first term, a_1, is 50. Use the formula $a_k=a_1+d(k-1)$ to find the explicit formula. $a_k=50+(-2)(k-1)$ $a_k=50-2k+2$ $a_k=52-2k$
$\sum_{k=1}^{11}52-2k$	The first term is 50 which is $52-2(1)$ so the lower limit is 1. The last term is 30 which is $52-2(11)$ so the upper limit is 11.

Practice Exercises A

Write out and evaluate each sum.

1. $\sum_{k=2}^{6} \frac{1}{k^2}$
2. $\sum_{k=1}^{5} 3k-2$
3. $\sum_{k=1}^{5} 2^k - 1$
4. $\sum_{k=1}^{4} (-1)^k k$
5. $\sum_{k=3}^{7} \frac{k}{k+2}$
6. $\sum_{k=3}^{10} 2k+1$

Write each series using sigma notation.

7. $5+7+9+...+17$
8. $6+5+4+...+(-1)$
9. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{20}$
10. $1+4+9+16+...+49$
11. $37+34+31+...+13$
12. $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+...+\frac{9}{64}$

Deriving the Formula for the Sum of a Finite Geometric Series

Exploration:

1. Multiply: $(x-1)(x+1)$
2. Multiply: $(x-1)(x^2+x+1)$
3. Multiply: $(x-1)(x^3+x^2+x+1)$
4. Multiply: $(x-1)(x^4+x^3+x^2+x+1)$
5. Without multiplying, determine the product of $(x-1)(x^{n-1}+x^{n-2}+...+x^2+x+1)$.

Answer:

1. $(x-1)(x+1)=x^2-1$
2. $(x-1)(x^2+x+1)=x^3-1$
3. $(x-1)(x^3+x^2+x+1)=x^4-1$
4. $(x-1)(x^4+x^3+x^2+x+1)=x^5-1$
5. $(x-1)(x^{n-1}+x^{n-2}+...+x^2+x+1)=x^n-1$

In general, a finite geometric series has the form $\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$. If you rewrite the sum of a geometric series in standard polynomial form (with the exponents in descending order), then you get: $ar^{n-1} + ... + ar^3 + ar^2 + ar + a$. Factoring out the first term you get: $a\left(r^{n-1} + ... + r^3 + r^2 + r + 1\right)$. Which looks a lot like $x^{n-1} + x^{n-2} + ... + x^2 + x + 1$. We know that $(x-1)\left(x^{n-1} + x^{n-2} + ... + x^2 + x + 1\right) = x^n - 1$. If we solve the equation for $x^{n-1} + x^{n-2} + ... + x^2 + x + 1$, then we get $x^{n-1} + x^{n-2} + ... + x^2 + x + 1 = \dfrac{x^n - 1}{x - 1}$. With a little bit of algebra we get $\dfrac{x^n - 1}{x - 1} = \dfrac{(-1)(1 - x^n)}{(-1)(1 - x)} = \dfrac{1 - x^n}{1 - x}$. Since $r^{n-1} + ... + r^3 + r^2 + r + 1$ looks like $x^{n-1} + x^{n-2} + ... + x^2 + x + 1$ we can conclude that $a\left(r^{n-1} + ... + r^3 + r^2 + r + 1\right) = a\left(\dfrac{1 - r^n}{1 - r}\right)$.

Sum of a Finite Geometric Series
The formula for the sum of a finite geometric series is $S_n = \dfrac{a(1 - r^n)}{1 - r}$ where *a* is the first term, *r* is the common ratio, and *n* is the number of terms.

Example 4:
Evaluate the sum of the finite geometric series.

a. $1 + 3 + 9 + 27 + 81 + 243 + 729$

b. $\sum_{k=1}^{10} 5\left(\dfrac{1}{2}\right)^{k-1}$

a. $1 + 3 + 9 + 27 + 81 + 243 + 729$	Use $S_n = \dfrac{a(1 - r^n)}{1 - r}$.
$S_7 = \dfrac{1(1 - 3^7)}{1 - 3}$ $S_7 = \dfrac{-2186}{-2}$ $S_7 = 1093$	The common ratio is 3. $r = 3$ The first term is 1. $a = 1$ There are 7 terms. $n = 7$

b. $\sum_{k=1}^{10}\left(\frac{1}{2}\right)^{k-1}$	Use $S_n = \frac{a\left(1-r^n\right)}{1-r}$.
$S_{10} = \frac{1\left(1-\frac{1}{2}^{10}\right)}{1-\frac{1}{2}}$ $S_{10} = \frac{\frac{1023}{1024}}{\frac{1}{2}}$ $S_{10} = \frac{2046}{1024} \approx 1.998$	The common ratio is $\frac{1}{2}$. $r = \frac{1}{2}$ The first term is $\left(\frac{1}{2}\right)^{1-1} = 1$. $a = 1$ The upper limit is 10. $n = 10$

Practice Exercises B

Evaluate the sum of the finite geometric series.

1. $1+4+16+...+4096$
2. $3+6+12+24+...+768$
3. $4-12+36-...-8748$
4. $81+27+...+\frac{1}{9}$
5. $1+\frac{1}{5}+\frac{1}{25}+...+\frac{1}{3125}$
6. $\frac{7}{10}-\frac{7}{100}+...+\frac{7}{100,000}$
7. $\sum_{k=1}^{5}(-2)(-3)^{k-1}$
8. $\sum_{k=1}^{6}(-1)(-5)^{k-1}$
9. $\sum_{k=1}^{5}(-2)(6)^{k-1}$
10. $\sum_{k=1}^{7}(-3)\left(\frac{1}{4}\right)^{k-1}$
11. $\sum_{k=1}^{6}3\left(\frac{1}{2}\right)^{k-1}$
12. $\sum_{k=1}^{7}4\left(\frac{2}{3}\right)^{k-1}$
13. A professional baseball player signs a contract with a beginning salary of $2,250,000 for the first year and an annual increase of 5% per year beginning in the second year. How much money in total will the athlete make if his contract is for 6 years? Round to the nearest dollar.
14. You are investigating two employment opportunities. Company A offers $33,000 the first year. During the next four years the salary is guaranteed to increase by 7% per year. Company B offers $35,000 the first year, with guaranteed annual increases of 4% per year after that. Which company offers the better total salary for a five-year contract?
15. A job starts at $28,700 with a yearly increase of 2.5% after the first year. The average person retires after 30 years. What is the total life-time salary over the 30-year period? Round to the nearest dollar.

Sum of an Infinite Geometric Series (Honors)

Calculus is required to derive the formula for the sum of an infinite geometric series. Informally, we can prove that $S=\dfrac{a}{1-r}$ if $|r|<1$ by looking at an example. The repeating decimal $0.\bar{3}$ can be written as a geometric series as follows $\sum_{k=1}^{n} 3(0.1)^k$. The first term, *a*, would be $3(0.1)^1 = 0.3$ and the common ratio, *r*, 0.1.

n	r^n	$a(1-r^n)$	$\dfrac{a(1-r^n)}{1-r}$
1	0.1	$0.3\left(1-(0.1)^1\right)=0.27$	$\dfrac{0.27}{1-0.1}=.3$
2	0.01	$0.3\left(1-(0.1)^2\right)=0.297$	$\dfrac{0.297}{1-0.1}=.33$
3	0.001	$0.3\left(1-(0.1)^3\right)=0.2997$	$\dfrac{0.2997}{1-0.1}=.333$
4	0.0001	$0.3\left(1-(0.1)^4\right)=0.29997$	$\dfrac{0.29997}{1-0.1}=.3333$
5	0.00001	$0.3\left(1-(0.1)^5\right)=0.299997$	$\dfrac{0.299997}{1-0.1}=.33333$
6	0.000001	$0.3\left(1-(0.1)^6\right)=0.2999997$	$\dfrac{0.2999997}{1-0.1}=.333333$
7	0.0000001	$0.3\left(1-(0.1)^7\right)=0.29999997$	$\dfrac{0.29999997}{1-0.1}=.3333333$
8	0.00000001	$0.3\left(1-(0.1)^8\right)=0.299999997$	$\dfrac{0.299999997}{1-0.1}=.33333333$
9	0.000000001	$0.3\left(1-(0.1)^9\right)=0.2999999997$	$\dfrac{0.2999999997}{1-0.1}=.333333333$
10	0.0000000001	$0.3\left(1-(0.1)^{10}\right)=0.29999999997$	$\dfrac{0.29999999997}{1-0.1}=.3333333333$

By looking at the values in the table you can see that as *n* gets infinitely larger r^n gets closer to zero, which makes $a(1-r^n)$ approach the value of *a*. Thus, the sum of an infinite geometric series for $|r|<1$ is $S=\dfrac{a}{1-r}$. When $|r|<1$ the geometric series has a finite sum and it is said that the series **converges.**

Through a similar process we can show that for $|r| \geq 1$ there is no finite sum. Consider the geometric series $\sum_{k=1}^{n} 2(1.01)^k$. The first term, *a*, is $2(1.01)^1 = 2.02$ and the common ratio, *r*, is 1.01.

n	r^n	$a(1-r^n)$	$\frac{a(1-r^n)}{1-r}$
1	1.01	$2.02\left(1-(1.01)^1\right) = -0.0202$	$\frac{-0.0202}{1-1.01} = 2.02$
2	1.0201	$2.02\left(1-(1.01)^2\right) = -0.0406$	$\frac{-0.0406}{1-1.01} = 4.0602$
3	1.0303	$2.02\left(1-(1.01)^3\right) = -0.0612$	$\frac{-0.0612}{1-1.01} = 6.120802$
4	1.0406	$2.02\left(1-(1.01)^4\right) = -0.082$	$\frac{-0.082}{1-1.01} = 8.20201002$
5	1.051	$2.02\left(1-(1.01)^5\right) = -0.103$	$\frac{-0.103}{1-1.01} = 10.3040301$
10	1.104622125	$2.02\left(1-(1.01)^{10}\right) \approx -0.2113367$	$\frac{-0.2113367}{1-1.01} \approx 21.13367$
100	2.704814	$2.02\left(1-(1.01)^{100}\right) \approx -3.443724$	$\frac{-3.443724}{1-1.01} \approx 344.3723955$

You can see that as *n* gets larger both r^n and $a(1-r^n)$ get larger, which makes the sum $S = \frac{a(1-r^n)}{1-r}$ grow without bound. Thus, the sum of an infinite geometric series for $|r| \geq 1$ is not a finite number. It is said that a geometric series with $|r| \geq 1$ **diverges** because there is no finite sum.

Sum of an Infinite Geometric Series

The formula for the sum of an infinite geometric series is $S = \frac{a}{1-r}$ where *a* is the first term and *r*, such that $|r| < 1$, is the common ratio. There is no finite sum for an infinite geometric series if $|r| \geq 1$.

Example 5:
Identify the common ratio. Then determine if the geometric series will converge or diverge.

a. $\frac{5}{6} + \frac{25}{36} + \frac{125}{216} + \ldots$

b. $\sum_{k=1}^{\infty} \left(\frac{4}{\pi}\right)^k$

a. $\frac{5}{6}+\frac{25}{36}+\frac{125}{216}+...$ The common ratio is $\frac{25}{36}\div\frac{5}{6}=\frac{25}{36}\cdot\frac{6}{5}=\frac{5}{6}$. Since $\left\|\frac{5}{6}\right\|<1$, the geometric series will converge.	b. $\sum_{k=1}^{\infty}\left(\frac{4}{\pi}\right)^k$ The common ratio is $\frac{4}{\pi}\approx 1.273$. Since $\left\|\frac{4}{\pi}\right\|\geq 1$, the geometric series will diverge.

Example 6:
Find the sum of the infinite geometric series.

a. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+...$

b. $\sum_{k=1}^{\infty}5\left(\frac{1}{\pi}\right)^k$

a. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+...$	
$S=\dfrac{\frac{1}{2}}{1-\left(-\frac{1}{2}\right)}=\dfrac{\frac{1}{2}}{\frac{2}{2}+\frac{1}{2}}$ $S=\dfrac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$	The common ratio is $-\frac{1}{4}\div\frac{1}{2}=-\frac{1}{4}\cdot 2=-\frac{1}{2}$ and $\left\|-\frac{1}{2}\right\|<1$. The first term is $\frac{1}{2}$. Use the formula $S=\frac{a}{1-r}$.

b. $\sum_{k=1}^{\infty}5\left(\frac{1}{\pi}\right)^k$	
$S=\dfrac{\frac{5}{\pi}}{1-\frac{1}{\pi}}=\dfrac{\frac{5}{\pi}}{\frac{\pi}{\pi}-\frac{1}{\pi}}$ $S=\dfrac{\frac{5}{\pi}}{\frac{\pi-1}{\pi}}=\frac{5}{\pi}\cdot\frac{\pi}{\pi-1}=\frac{5}{\pi-1}$	The common ratio is $\frac{1}{\pi}$ and $\left\|\frac{1}{\pi}\right\|<1$. The first term is $5\left(\frac{1}{\pi}\right)^1=\frac{5}{\pi}$. Use the formula $S=\frac{a}{1-r}$.

Practice Exercises C

Identify the common ratio. Then determine if the geometric series will converge or diverge.

1. $0.0004-0.004+0.04-...$

2. $-\frac{1}{24}+\frac{1}{12}-\frac{1}{6}+\frac{1}{3}+...$

3. $3+\frac{3}{4}+\frac{3}{4^2}+\frac{3}{4^3}+...$

4. $\sum_{k=1}^{\infty}47\left(0.01\right)^k$

5. $\sum_{k=1}^{\infty}\frac{1}{2}\left(1.02\right)^{k-1}$

6. $\sum_{k=1}^{\infty}18\left(\frac{1}{3}\right)^{k-1}$

Find the sum of each infinite geometric series.

7. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...$

8. $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+...$

9. $2-\frac{2}{5}+\frac{2}{25}-\frac{2}{125}+...$

10. $1024+128+16+2+...$

11. $12+6+3+\frac{3}{2}+...$

12. $1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+...$

13. $\sum_{k=1}^{\infty}3\left(0.1\right)^{k-1}$

14. $\sum_{k=1}^{\infty}\left(\frac{2}{\pi}\right)^k$

15. $\sum_{k=1}^{\infty}\left(\frac{e}{3}\right)^{k-1}$

16. $\sum_{k=1}^{\infty}83\left(\frac{1}{100}\right)^k$

17. $\sum_{k=1}^{\infty}-2\left(0.6\right)^{k-1}$

18. $\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{k-1}$

19. The height a ball bounces is less than the height of the previous bounce due to friction. Suppose a ball is dropped from a height of 4 feet and rebounds to 98% of the height of the previous bounce. Write the series in sigma notation. What is the total vertical distance traveled by the ball when it comes to rest?

20. Because of friction and air resistance, each swing of a pendulum is a little shorter than the previous one. Suppose the first swing of a pendulum has a length of 5 inches and the return swing is 4.8 inches. Write the series in sigma notation. What is the total distance traveled by the pendulum when it comes to rest?

Deriving the Sum of a Finite Arithmetic Series (Honors)

VOCABULARY

An **arithmetic series** is the sum of the terms of an arithmetic sequence. An arithmetic sequence can be written explicitly using the formula $a_n = a_1 + (n-1)d$, where a_1 is the first term and *d* is the common difference. It can also be written recursively using the formula $a_n = a_{n-1} + d$, where a_{n-1} is the previous term and *d* is the common difference.

The sum of the first *n* terms of an arithmetic sequence would be: $S_n = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$. Using the explicit formula for the *n*th term the sum can be rewritten as $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_1 + (n-2)d) + (a_1 + (n-1)d)$. The sum of the first *n* terms could also be written as $S_n = a_n + a_{n-1} + ... + a_3 + a_2 + a_1$. Using the explicit formula for the *n*th term the sum can be rewritten as $S_n = a_n + (a_n - d) + (a_n - 2d) + ... + (a_n - (n-2)d) + (a_n - (n-1)d)$. If you were to add the two equations together then you would get:

$$\begin{array}{l} S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_1 + (n-2)d) + (a_1 + (n-1)d) \\ \underline{+S_n = a_n + (a_n - d) + (a_n - 2d) + ... + (a_n - (n-2)d) + (a_n - (n-1)d)} \\ 2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + ... + (a_1 + a_n) + (a_1 + a_n) \end{array}$$

This result can be simplified to $2S_n = n(a_1 + a_n)$ since there are *n* sums of $a_1 + a_n$. Solving the equation for S_n produces $S_n = \frac{n}{2}(a_1 + a_n)$. If you were to substitute the explicit formula in for a_n then the result would be $S_n = \frac{n}{2}\left[a_1 + a_1 + (n-1)d\right] = \frac{n}{2}\left[2a_1 + (n-1)d\right]$. This can also be used to find the sum of an arithmetic series.

The Sum of a Finite Arithmetic Series

The formula for the sum of a finite arithmetic series is $S_n = \frac{n}{2}(a_1 + a_n)$, where a_1 is the first term, a_n is the last term, and *n* is the number of terms.

The sum of a finite arithmetic series can also be found using the formula $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$, where a_1 is the first term, *d* is the common difference and *n* is the number of terms.

Example 7:
Find the sum of the finite arithmetic series.

a. $-5-11-17-23-...-71$

b. $\sum_{k=1}^{8} 4k+3$

a. $-5-11-17-23-...-71$	
$S_{12}=\frac{12}{2}(-5+-71)$ $S_{12}=6(-76)$ $S_{12}=-456$	Use the formula $S_n=\frac{n}{2}(a_1+a_n)$. The common difference is $-11-(-5)=-6$. The first term is -5. The last term is -71. The number of terms can be found using the explicit formula $a_n=a_1+(n-1)d$. $-71=-5+(n-1)(-6)$ $-71=-5-6n+6$ $-71=1-6n$ $-72=-6n$ $12=n$

b. $\sum_{k=1}^{8} 4k+3$	
$S_8=\frac{8}{2}(7+35)$ $S_8=4(42)$ $S_8=168$	Use the formula $S_n=\frac{n}{2}(a_1+a_n)$. The common difference is 4. The first term is $4(1)+3=7$. The last term is $4(8)+3=35$. The number of terms is 8.

Practice Exercises D

Find the sum of the finite arithmetic series.

1. $8+13+18+23+...+88$
2. $2+8+14+20+...+116$
3. $2+(-1)+(-4)+...+(-40)$
4. $4+2+0+....+(-20)$
5. $7+19+31+43+...+115$
6. $1+5+9+13+...+45$
7. $\sum_{k=1}^{25}(4k-14)$
8. $\sum_{k=1}^{20}(6k-21)$
9. $\sum_{k=1}^{15}(11-5k)$
10. $\sum_{k=1}^{17}(10-4k)$
11. $\sum_{k=1}^{14}(3k-1)$
12. $\sum_{k=3}^{22}(2k-1)$
13. A theater has 20 seats in the first row, 22 seats in the second row, increasing by 2 seats per row for a total of 25 rows.

    a. Write an arithmetic series to represent the number of seats in the theater.

    b. Find the total seating capacity of the theater.

    c. If tickets are $9.25 per seat, how much money will the theater make if the theater is filled to capacity?

14. A supermarket displays cans in a triangle. There are 15 cans in the bottom row and each successive row has one fewer can than the previous row for a total of 14 rows.

    a. Use summation notation to write the series for the triangle.

    b. How many cans are in the display?

15. A company offers a starting yearly salary of $28,500 with raises of $1,000 each year after the first year. Find the total salary over a 15-year period.

Rational and Radical Expressions and Equations

Unit 2 Cluster 7 (A.APR.6 and A.APR.7): Rational Expressions

Cluster 7: Rewrite rational expressions

2.7 Rewrite simple rational expressions in different forms using inspection, long division, or, for the more complicated examples, a computer algebra system (CAS).

2.7 Add, subtract, multiply, and divide rational expressions.

2.7 Closure of rational expressions under addition, subtraction, multiplication, and division by a nonzero rational expression.

VOCABULARY

A **rational function** is a function of the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x)\neq 0$. For example, $f(x)=\frac{3x-4}{x+1}$ is a rational function.

Example 1:

Simplify $f(x)=\frac{-5x^3+4x^2+6x}{x}$

$f(x)=\frac{-5x^3+4x^2+6x}{x}$	$x\neq 0$ because the denominator must be a nonzero polynomial
$f(x)=\frac{-5x^3}{x}+\frac{4x^2}{x}+\frac{6x}{x}$	Rewrite the rational expression as the sum of fractions with a common denominator.
$f(x)=-5x^2+4x+6$	Simplify

Example 2:

Simplify $f(x)=\frac{x^2-4}{x+2}$

$f(x)=\frac{x^2-4}{x+2}$	$x\neq -2$ because the denominator must be a nonzero polynomial
$f(x)=\frac{(x+2)(x-2)}{x+2}$	Factor
$f(x)=\frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$	Simplify like terms
$f(x)=x-2$	

Example 3:

Simplify $f(x)=\dfrac{x^2-5x+4}{x^2+2x-3}$

$f(x)=\dfrac{x^2-5x+4}{x^2+2x-3}$	
$f(x)=\dfrac{(x-4)(x-1)}{(x+3)(x-1)}$	Factor $x\neq -3$ or $x\neq 1$ because the denominator must be a nonzero polynomial
$f(x)=\dfrac{(x-4)\cancel{(x-1)}}{(x+3)\cancel{(x-1)}}$	Simplify like terms
$f(x)=\dfrac{(x-4)}{(x+3)}$	

Example 4:

Simplify $f(x)=\dfrac{x^2-8x+15}{x^3-125}$

$f(x)=\dfrac{x^2-8x+15}{x^3-125}$	
$f(x)=\dfrac{(x-3)(x-5)}{(x-5)(x^2+5x+25)}$	Factor $x\neq 5$ because the denominator must be a nonzero polynomial
$f(x)=\dfrac{(x-3)\cancel{(x-5)}}{\cancel{(x-5)}(x^2+5x+25)}$	Simplify like terms
$f(x)=\dfrac{(x-3)}{(x^2+5x+25)}$	

Practice Exercises A

Simplify each rational expression.

1. $\dfrac{8x^3-4x^2-2x}{2x}$
2. $\dfrac{-4x^3-10x^2+2x}{2}$
3. $\dfrac{3x^2+4x+9}{x}$
4. $\dfrac{x^2-8x-20}{x-10}$
5. $\dfrac{x^2-4x-5}{x+1}$
6. $\dfrac{x-9}{x^2-18x+81}$
7. $\dfrac{6x^2-47x-8}{x-8}$
8. $\dfrac{3x^2+25x+42}{3x+7}$
9. $\dfrac{2x+7}{4x^2-49}$

10. $\dfrac{x^2-4}{x^2+4x+4}$	11. $\dfrac{x^2+16x+63}{x^2+3x-54}$	12. $\dfrac{x^2+9x+8}{x^2+16x+64}$
13. $\dfrac{2x^2-13x-7}{2x^2+21x+10}$	14. $\dfrac{25x^2-4}{5x^2+8x-4}$	15. $\dfrac{3x^2+25x-18}{x^2+10x+9}$
16. $\dfrac{x^3-1}{x-1}$	17. $\dfrac{x+2}{x^3+8}$	18. $\dfrac{x^3+216}{x+6}$
19. $\dfrac{x^2-16}{x^3+64}$	20. $\dfrac{27x^3-8}{3x^2+16x-12}$	21. $\dfrac{4x^2-2x+1}{8x^3+1}$

Long Division with Polynomials

A rational expression $f(x)=\dfrac{p(x)}{q(x)}$ can be thought of as $p(x)$ divided by $q(x)$. Division of polynomials, like division of real numbers, uses multiplication and subtraction.

Let us review the algorithm for long division with real numbers. Consider 335 divided by 23.

$335 \div 23$	335 is the dividend 23 is the divisor
$23\overline{)335}$	Rewrite the division problem so that the dividend is under the long division symbol and the divisor is on the outside
$\begin{array}{r} 1 \\ 23\overline{)335} \\ \underline{-23} \\ 105 \end{array}$	Divide 33 by 23. $\dfrac{33}{23}$ is a little over one. This becomes the first term of your quotient. Multiply 23 by 1 and subtract the product from the dividend. Bring down the next unused term. This is your new dividend.
$\begin{array}{r} 14 \\ 23\overline{)335} \\ \underline{-23} \\ 105 \\ \underline{-92} \\ 13 \end{array}$	Divide 105 by 23. $\dfrac{105}{23}$ is a little over four. This becomes the next term of your quotient. Multiply 23 by 4 and subtract the product from the dividend. The remainder is 13. Therefore, $335 \div 23 = 14 + \dfrac{13}{23}$

Example 5:

Simplify using long division $\dfrac{2x^2-5x-12}{x-4}$

$\dfrac{2x^2-5x-12}{x-4}$	$2x^2-5x-12$ is the dividend $x-4$ is the divisor
$x-4\overline{)2x^2-5x-12}$	Rewrite the rational expression with the dividend under the long division symbol and the divisor on the outside.
$\begin{array}{r} 2x \\ x-4\overline{)\;2x^2-5x-12} \\ -\left(2x^2-8x\right) \\ \hline 3x-12 \end{array}$	Divide $2x^2$ by x. $\dfrac{2x^2}{x}=2x$ this becomes the first term of your quotient. Multiply $x-4$ by $2x$ and subtract the product from the dividend. Bring down the next unused term. This is your new dividend.
$\begin{array}{r} 2x+3 \\ x-4\overline{)\;2x^2-5x-12} \\ -\left(2x^2-8x\right) \\ \hline 3x-12 \\ -\left(3x-12\right) \\ \hline 0 \end{array}$	Divide $3x$ by x. $\dfrac{3x}{x}=3$ this becomes the next term of your quotient. Multiply $x-4$ by 3 and subtract the product from the dividend. The remainder is zero. Therefore, $\dfrac{2x^2-5x-12}{x-4}=2x+3$

Example 6:

Simplify using long division $\dfrac{3x^3+5x^2+8x+7}{3x+2}$

$\dfrac{3x^3+5x^2+8x+7}{3x+2}$	$3x^3+5x^2+8x+7$ is the dividend $3x+2$ is the divisor
$3x+2\overline{)3x^3+5x^2+8x+7}$	Rewrite the rational expression with the dividend under the long division symbol and the divisor on the outside.
$\begin{array}{r} x^2 \\ 3x+2\overline{)\;3x^3+5x^2+8x+7} \\ -\left(3x^3+2x^2\right) \\ \hline 3x^2+8x \end{array}$	Divide $3x^3$ by $3x$. $\dfrac{3x^3}{3x}=x^2$ this becomes the first term of your quotient. Multiply $3x+2$ by x^2 and subtract the product from the dividend. Bring down the next unused term. This is your new dividend.

$$\begin{array}{r} x^2 + x \\ 3x+2 \overline{\big) \; 3x^3 + 5x^2 + 8x + 7} \\ -(3x^3 + 2x^2) \\ \hline 3x^2 + 8x \\ -(3x^2 - 2x) \\ \hline 6x + 7 \end{array}$$	Divide $3x^2$ by $3x$. $\frac{3x^2}{3x} = x$ this becomes the next term of your quotient. Multiply $3x+2$ by x and subtract the product from the dividend. Bring down the next unused term. This is your new dividend**.**
$$\begin{array}{r} x^2 + x + 2 \\ 3x+2 \overline{\big) \; 3x^3 + 5x^2 + 8x + 7} \\ -(3x^3 + 2x^2) \\ \hline 3x^2 + 8x \\ -(3x^2 - 2x) \\ \hline 6x + 7 \\ -(6x + 4) \\ \hline 3 \end{array}$$	Divide $6x$ by $3x$. $\frac{6x}{3x} = 2$ this becomes the next term of your quotient. Multiply $3x+2$ by 2 and subtract the product from the dividend. The remainder is 3. Therefore, $\frac{3x^3+5x^2+8x+7}{3x+2} = x^2 + x + 2 + \frac{3}{3x+2}$

Practice Exercises B

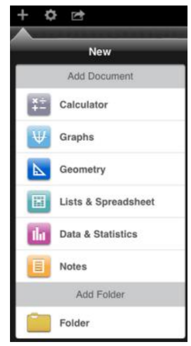
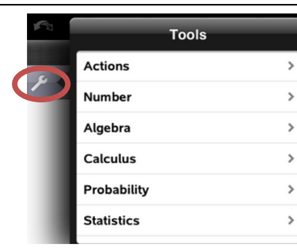
Simplify using long division.

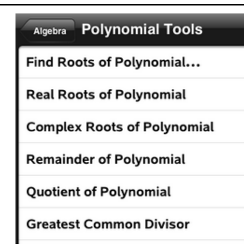
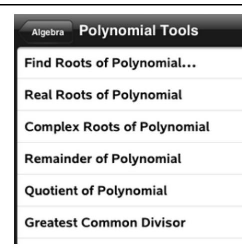
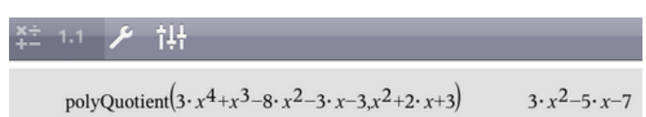
1. $\frac{x^2-10x-25}{x-5}$	2. $\frac{x^2-8x-16}{x+4}$	3. $\frac{x^2-9x+21}{x-4}$
4. $\frac{3x^2-11x+33}{x-6}$	5. $\frac{4x^2-2x+3}{x-1}$	6. $\frac{10x^2-6x+3}{5x+2}$
7. $\frac{6x^2-7x+3}{3x-2}$	8. $\frac{x^3-2x^2+4x-5}{x+3}$	9. $\frac{x^3-1}{x+1}$
10. $\frac{x^3+4x^2+7x-9}{x+3}$	11. $\frac{2x^3+3x^2-x-3}{x+2}$	12. $\frac{10x^3+6x^2-9x+10}{5x-2}$
13. $\frac{6x^3-11x^2+11x-2}{2x-3}$	14. $\frac{4x^3-8x^2+3x-1}{2x+1}$	15. $\frac{3x^3-5x^2-3x-2}{x-2}$

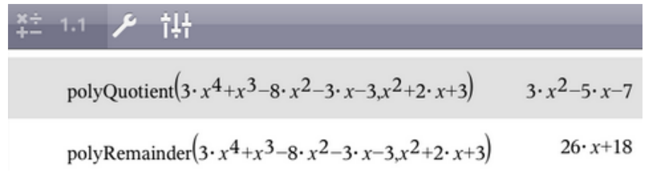
Using Technology to Divide Polynomials

For more complicated polynomial division you may want to use a computer algebra system such as the TI-Nspire CAS. You can download an app of the TI-Nspire CAS for your iPad or you can purchase a TI-Nspire CAS calculator. The instructions below are for the iPad app version.

Example 7: Divide $3x^4+x^3-8x^2-3x-3$ by x^2+2x+3

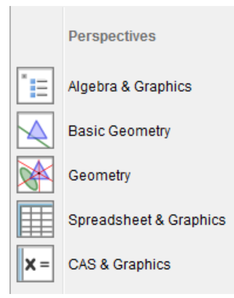
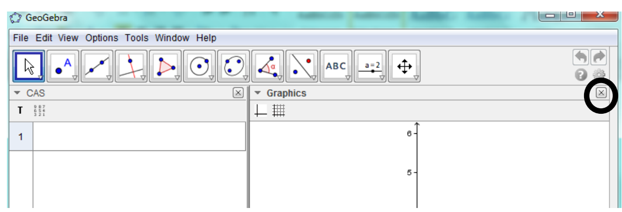
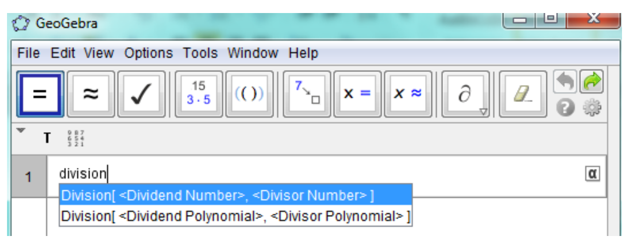
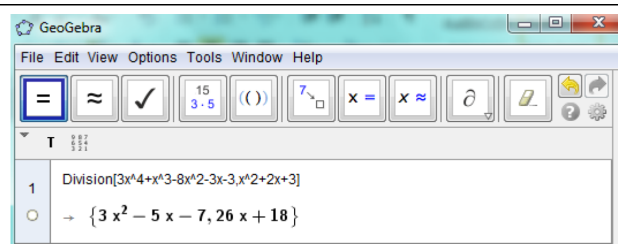
<table>
<tr>
<td>Create a new document by pushing the + symbol in the upper left hand corner. A menu, like the one at the right, will appear. Select Calculator.</td>
<td>
</td>
</tr>
<tr>
<td>You will have a document that you can type mathematical equations in.</td>
<td>
</td>
</tr>
<tr>
<td>Push the wrench at the top of the screen to bring up the Tools menu then select Algebra. A new Algebra menu will appear. Scroll down and Select Polynomial Tools.</td>
<td>


</td>
</tr>
<tr>
<td>The Polynomial Tools menu should have Remainder of Polynomial and Quotient of Polynomial. You will need both of these tools to divide.</td>
<td>
</td>
</tr>
<tr>
<td>Select Quotient of Polynomial. Enter the dividend polynomial (for this example it is $3x^4+x^3-8x^2-3x-3$) then a comma and the divisor polynomial (for this example it is x^2+2x+3). Press Enter and you will have the quotient, but not the remainder if there is one.</td>
<td>
</td>
</tr>
</table>

Push the wrench to bring up the Polynomial Tools menu again. Select Remainder of Polynomial. Enter the dividend polynomial (for this example it is $3x^4+x^3-8x^2-3x-3$) then a comma and the divisor polynomial (for this example it is x^2+2x+3). Press Enter and you will have the remainder.	
$3x^4+x^3-8x^2-3x-3$ divided by x^2+2x+3 is $3x^2-5x-7+\dfrac{26x+18}{x^2+2x+3}$.	

You can also use Geogebra as a computer algebra system. The program is free and can be downloaded to your computer at www.geogebra.org. (Instructions for PC version.)

Example 7B: Divide $3x^4+x^3-8x^2-3x-3$ by x^2+2x+3

When you open Geogebra you should see a menu like the one at the right. Select CAS & Graphics. If it there is no menu, then press ctrl + shift + k to bring up the CAS screen.	
You should see a screen like the one at the right. To close the Graphics screen click on the x in the right hand corner.	
To the right of the number 1, start typing division and some options will come up. Select Division [<Dividend Polynomial>, <Divisor Polynomial>].	
The dividend polynomial is $3x^4+x^3-8x^2-3x-3$ and the divisor polynomial is x^2+2x+3. Once you have entered the polynomials, press enter and you will get an answer in the form {Quotient,Remainder}.	
$3x^4+x^3-8x^2-3x-3$ divided by x^2+2x+3 is $3x^2-5x-7+\dfrac{26x+18}{x^2+2x+3}$.	

Practice Exercises C

Use a computer algebra system to divide the polynomials.

1. $\dfrac{-5x^4 - x^3 + 31x^2 - 31x - 6}{-5x^2 + 9x - 2}$
2. $\dfrac{3x^4 - x^3 - 8x^2 + 5x - 4}{-3x^2 + x - 7}$
3. $\dfrac{x^4 + 8x^3 + 17x^2 + 6x - 13}{x^2 + 5x + 2}$
4. $\dfrac{-2x^4 - 20x^3 + 6x^2 + 20x + 3}{x^2 + 10x - 2}$
5. $\dfrac{8x^4 - 2x^3 - 18x^2 - 5}{4x^2 - x - 9}$
6. $\dfrac{-7x^5 - 3x^3 + 63x^2 + 7}{x^3 - 9}$

Multiplying and Dividing Rational Expressions

VOCABULARY

If the numerator and denominator of a rational expression have no common factors, other than ± 1, then the rational expression is in **simplified form**.

To multiply a rational expression by another, multiply the numerator with the numerator and the denominator with the denominator.

$\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd} \longleftarrow$ Simplify $\dfrac{ac}{bd}$ if possible ($b \neq 0$ and $d \neq 0$)

To divide one rational expression by another, multiply the first expression by the reciprocal of the second expression.

$\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc} \longleftarrow$ Simplify $\dfrac{ad}{bc}$ if possible ($b \neq 0$, $d \neq 0$, and $c \neq 0$)

Example 8:

Multiply $\dfrac{3x^2y}{2z^2} \cdot \dfrac{8x^3z}{15y^3}$

$\dfrac{3x^2y}{2z^2} \cdot \dfrac{8x^3z}{15y^3}$	
$\dfrac{24x^5yz}{30z^2y^3}$	Multiply the numerator with the numerator and the denominator with the denominator.
$\dfrac{24}{30} \cdot \dfrac{x^5}{1} \cdot \dfrac{y}{y^3} \cdot \dfrac{z}{z^2}$	Separate like terms

$\frac{4}{5}\cdot\frac{x^5}{1}\cdot\frac{1}{y^2}\cdot\frac{1}{z}$	Simplify using integer exponent properties
$\frac{4x^5}{5y^2z}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 9:

Multiply $\frac{x+1}{x+4}\cdot\frac{x^2+4x}{x+2}$

$\frac{x+1}{x+4}\cdot\frac{x^2+4x}{x+2}$	
$\frac{x+1}{x+4}\cdot\frac{x(x+4)}{x+2}$	Factor
$\frac{x+1}{\cancel{x+4}}\cdot\frac{x\cancel{(x+4)}}{x+2}$	Identify the like factors.
$\frac{x+1}{1}\cdot\frac{x}{x+2}$	Simplify the like factors.
$\frac{x(x+1)}{x+2}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 10:

Multiply $\frac{x^2-4}{2x+2}\cdot\frac{x^2-2x-3}{x^2+4x+4}$

$\frac{x^2-4}{2x+2}\cdot\frac{x^2-2x-3}{x^2+4x+4}$	
$\frac{(x-2)(x+2)}{2(x+1)}\cdot\frac{(x-3)(x+1)}{(x+2)(x+2)}$	Factor
$\frac{(x-2)\cancel{(x+2)}}{2\cancel{(x+1)}}\cdot\frac{(x-3)\cancel{(x+1)}}{\cancel{(x+2)}(x+2)}$	Identify the like factors.
$\frac{(x-2)}{2}\cdot\frac{(x-3)}{(x+2)}$	Simplify the like factors.
$\frac{(x-2)(x-3)}{2(x+2)}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 11:

Divide $\dfrac{3x^6yz^2}{7xy^3} \div \dfrac{15xy^3z^8}{7x^6y^2z^6}$

$\dfrac{3x^6yz^2}{7xy^3} \div \dfrac{15xy^3z^8}{7x^6y^2z^6}$	
$\dfrac{3x^6yz^2}{7xy^3} \cdot \dfrac{7x^6y^2z^6}{15xy^3z^8}$	Multiply by the reciprocal of the term that follows the division symbol.
$\dfrac{21x^{12}y^3z^8}{105x^2y^6z^8}$	Multiply the numerator with the numerator and the denominator with the denominator.
$\dfrac{21}{105} \cdot \dfrac{x^{12}}{x^2} \cdot \dfrac{y^3}{y^6} \cdot \dfrac{z^8}{z^8}$	Separate like terms
$\dfrac{1}{5} \cdot \dfrac{x^{10}}{1} \cdot \dfrac{1}{y^3} \cdot \dfrac{1}{1}$	Simplify using integer exponent properties
$\dfrac{x^{10}}{5y^3}$	Multiply the numerator with the numerator and the denominator with the denominator.

Example 12:

Divide $\dfrac{12x-20}{x^2-4x-21} \div \dfrac{9x^2-25}{3x^2+14x+15}$

$\dfrac{12x-20}{x^2-4x-21} \div \dfrac{9x^2-25}{3x^2+14x+15}$	
$\dfrac{x^2-4x-21}{12x-20} \cdot \dfrac{3x^2+14x+15}{9x^2-25}$	
$\dfrac{4(3x-5)}{(x-7)(x+3)} \cdot \dfrac{(x+3)(3x+5)}{(3x-5)(3x+5)}$	Factor
$\dfrac{4\cancel{(3x-5)}}{(x-7)\cancel{(x+3)}} \cdot \dfrac{\cancel{(x+3)}\cancel{(3x+5)}}{\cancel{(3x-5)}\cancel{(3x+5)}}$	Identify the like factors.
$\dfrac{4}{x-7} \cdot \dfrac{1}{1}$	Simplify the like factors.
$\dfrac{4}{x-7}$	Multiply the numerator with the numerator and the denominator with the denominator.

Practice Exercises D

Perform the indicated operation, if possible, simplify. Determine if your answer is a rational expression.

1. $\dfrac{8x^2}{9y}\cdot\dfrac{3y^2}{2x^5}$

2. $\dfrac{4x^6}{3y^7}\cdot\dfrac{9y^2}{2x^3}$

3. $\dfrac{-4x^3}{y^4}\div\dfrac{-2}{x^2y^4}$

4. $\dfrac{-1}{y^4z^3}\div\dfrac{6x^2y}{z^2}$

5. $\dfrac{6y^2}{5x^2}\div\dfrac{3y^2}{4x^6}$

6. $\dfrac{8-x}{6x-18}\cdot\dfrac{3x-9}{2x-16}$

7. $\dfrac{x+5}{x-6}\cdot\dfrac{2x-12}{x^2-25}$

8. $\dfrac{x+2}{x-6}\cdot\dfrac{3x^2}{x^2+4x+4}$

9. $\dfrac{x^2+5x-14}{3x^3-6x^2}\cdot\dfrac{2x^2+6x}{x^2+10x+21}$

10. $\dfrac{x^2-2x}{x^2-1}\cdot\dfrac{4x-4}{x^2-4}$

11. $\dfrac{x^2-2x-24}{4x^2+13x-12}\cdot\dfrac{8x-6}{x^2-6x}$

12. $\dfrac{2x^2+19x-10}{x^2+x-12}\cdot\dfrac{x^2-16}{x^2-100}$

13. $\dfrac{x^2+x-6}{x^2+5x+4}\cdot\dfrac{3x^2+14x+8}{2x^2+7x+3}$

14. $\dfrac{4x-24}{x^2-6x+5}\div\dfrac{-6x+36}{x^2-8x+15}$

15. $\dfrac{x+4}{x^2-36}\div\dfrac{4x^2+16x}{x^2-4x-12}$

16. $\dfrac{x^2-9}{2x-2}\div\dfrac{x^2-2x-3}{x-1}$

17. $\dfrac{5x^2+5x}{x-4}\div\dfrac{x^2-4x-5}{x^3-4x^2}$

18. $\dfrac{2x^2+9x+9}{x^2-8x+12}\div\dfrac{4x^2-9}{x^2-6x}$

19. $\dfrac{x^2+3x+2}{3x-18}\div\dfrac{x^2-1}{x^2-x-30}$

20. $\dfrac{3x^2+17x+10}{x^2+5x+6}\div\dfrac{3x+15}{x^2+2x}$

21. $\dfrac{x^3-64}{x^3+64}\div\dfrac{x^2-16}{x^2-4x+16}$

Simplify.

22. $\dfrac{x-1}{x^2-x-6}\div\dfrac{x^2+4x-5}{x^2+8x+12}\cdot\dfrac{2x+10}{x+6}$

23. $\dfrac{2x^2+3x}{x^2-16}\cdot\dfrac{25x^2-9}{4x^2+12x+9}\div\dfrac{25x+15}{2x^2+11x+12}$

24. $\dfrac{4x^2-9}{8x^3-27}\cdot\dfrac{4x^2+6x+9}{4x^2-8x+3}\div\dfrac{4x+6}{3x-9}$

25. $\dfrac{15x^2+5x-50}{32x^2-18}\div\dfrac{x^2-5x-14}{4x^2+9x-9}\cdot\dfrac{6x-42}{3x^2+4x-15}$

YOU DECIDE

Are rational expressions closed under multiplication and division? Justify your conclusion using the method of your choice.

Adding and Subtracting Rational Expressions

VOCABULARY To add a rational expression to another, find a common denominator then add the numerators. $\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}=\frac{ad}{bd}+\frac{cb}{bd}=\frac{ad+cb}{bd} \longleftarrow$ Simplify $\frac{ad+cb}{bd}$ if possible ($b\neq 0$ and $d\neq 0$) To subtract one rational expression from another, find a common denominator then subtract the numerators. $\frac{a}{b}-\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}-\frac{c}{d}\cdot\frac{b}{b}=\frac{ad}{bd}-\frac{cb}{bd}=\frac{ad-cb}{bd} \longleftarrow$ Simplify $\frac{ad-cb}{bd}$ if possible ($b\neq 0$ and $d\neq 0$) The **least common multiple** (LCM) for expressions is the smallest (non-zero) expression that is a multiple of two or more expressions.

Let us review how to add rational numbers. Add $\frac{1}{6}$ and $\frac{2}{15}$.

$\frac{1}{6}+\frac{2}{15}$	You need to find a common denominator. The least common multiple of 6 and 15 is 30. This will be the common denominator.
$\frac{1}{6}\cdot\frac{5}{5}+\frac{2}{15}\cdot\frac{2}{2}$	$30\div 6=5$ so multiply $\frac{1}{6}$ by $\frac{5}{5}$. $30\div 15=2$ so multiply $\frac{2}{15}$ by $\frac{2}{2}$.
$\frac{5}{30}+\frac{4}{30}=\frac{9}{30}$	Add the numerators.
$\frac{9}{30}=\frac{3}{10}$	Simplify.

Adding and subtracting rational expressions is similar to adding and subtracting rational numbers. The key is finding the common denominator or the least common multiple of the denominators.

Example 13:
Find the least common multiple for the expressions.

a. $10x^2y$ and $12y^3$ b. $x+1$ and x^2-1 c. $x^2+11x+24$ and $x^2+15x+56$

a. $10x^2y$ and $12y^3$	
$10x^2y = 2\cdot 5\cdot x\cdot x\cdot y$ $12y^3 = 2\cdot 2\cdot 3\cdot y\cdot y\cdot y$	Find the prime factorizations of each expression.
$10x^2y = 2\cdot 5\cdot x\cdot x\cdot y$ $12y^3 = 2\cdot 2\cdot 3\cdot y\cdot y\cdot y$	Identify what factors the expressions have in common $(2y)$. Then identify what is unique to both expressions $(5x^2$ and $6y^2)$.
$(2y)(5x^2)(6y^2) = 60x^2y^3$	Multiply the common factors and the unique factors to obtain the least common multiple.

b. $x+1$ and x^2-1	
$x-1 = (x-1)$ $x^2-1 = (x-1)(x+1)$	Factor each expression. Identify what factors the expressions have in common $(x+1)$. Then identify what is unique to both expressions $(x-1)$.
$(x+1)(x-1) = x^2-1$	Multiply the common factors and the unique factors to obtain the least common multiple.

c. $x^2+11x+24$ and $x^2+15x+56$	
$x^2+11x+24 = (x+3)(x+8)$ $x^2+15x+56 = (x+7)(x+8)$	Factor each expression. Identify what factors the expressions have in common $(x+8)$. Then identify what is unique to both expressions $(x+3)$ and $(x+7)$.
$(x+8)(x+3)(x+7) = x^3+18x^2+101x+168$	Multiply the common factors and the unique factors to obtain the least common multiple.

Example 14:
Perform the indicated operation. If possible, simplify.

a. $\frac{3}{7x}+\frac{4}{7x}$ b. $\frac{x-3}{x-5}-\frac{7-x}{x-5}$ c. $\frac{x^2}{x+2}-\frac{4}{x+2}$

a. $\frac{3}{7x}+\frac{4}{7x}$	
$\frac{3+4}{7x} = \frac{7}{7x}$	Add the numerators.
$\frac{7}{7x} = \frac{1}{x}$	Simplify.

b. $\frac{x-3}{x-5}-\frac{7-x}{x-5}$	
$\frac{(x-3)-(7-x)}{x-5}=\frac{2x-10}{x-5}$	Subtract the numerators.
$\frac{2x-10}{x-5}=\frac{2(x-5)}{x-5}=2$	Simplify.

c. $\frac{x^2}{x+2}-\frac{4}{x+2}$	
$\frac{x^2-4}{x+2}$	Subtract the numerators.
$\frac{x^2-4}{x+2}=\frac{(x-2)(x+2)}{x+2}=x-2$	Simplify.

Example 15:
Perform the indicated operations. If possible, simplify.

a. $\frac{1}{x+4}+\frac{8}{x^2-16}$ b. $\frac{3}{x+1}-\frac{2}{x-3}$ c. $\frac{x+2}{x^2-5x+4}-\frac{x}{x^2-3x-4}$

a. $\frac{1}{x+4}+\frac{8}{x^2-16}$	
$\frac{1}{x+4}+\frac{8}{(x+4)(x-4)}$	Factor and determine the LCM. The LCM is $(x+4)(x-4)$.
$\frac{1}{x+4}\cdot\frac{x-4}{x-4}+\frac{8}{(x+4)(x-4)}\cdot\frac{1}{1}$ $\frac{x-4}{(x+4)(x-4)}+\frac{8}{(x+4)(x-4)}$	$\frac{(x+4)(x-4)}{x+4}=x-4$ Multiply the first expression by $\frac{x-4}{x-4}$. $\frac{(x+4)(x-4)}{(x+4)(x-4)}=1$ Multiply the second expression by $\frac{1}{1}$.
$\frac{x-4+8}{(x+4)(x-4)}=\frac{x+4}{(x+4)(x-4)}$	Add the numerators.
$\frac{x+4}{(x+4)(x-4)}=\frac{1}{x-4}$	Simplify.
$\frac{1}{x+4}+\frac{8}{x^2-16}=\frac{1}{x-4}$	

b. $\frac{3}{x+1}-\frac{2}{x-3}$	
$\frac{3}{x+1}-\frac{2}{x-3}$	There are no common factors so the LCM is $(x+1)(x-3)$.
$\frac{3}{x+1}\cdot\frac{x-3}{x-3}-\frac{2}{x-3}\cdot\frac{x+1}{x+1}$ $\frac{3(x-3)}{(x+1)(x-3)}-\frac{2(x+1)}{(x-3)(x+1)}$	$\frac{(x+1)(x-3)}{x+1}=x-3$ Multiply the first expression by $\frac{x-3}{x-3}$. $\frac{(x+1)(x-3)}{x-3}=x+1$ Multiply the second expression by $\frac{x+1}{x+1}$.
$\frac{3x-9-(2x+2)}{(x+1)(x-3)}=\frac{3x-9-2x-2}{(x+1)(x-3)}$	Subtract the numerators.
$\frac{x-11}{(x+1)(x-3)}$	Simplify.
$\frac{3}{x+1}-\frac{2}{x-3}=\frac{x-11}{(x+1)(x-3)}$	

c. $\frac{x+2}{x^2-5x+4}-\frac{x}{x^2-3x-4}$	
$\frac{x+2}{(x-1)(x-4)}-\frac{x}{(x+1)(x-4)}$	Factor and determine the LCM. The LCM is $(x-4)(x-1)(x+1)$.
$\frac{x+2}{(x-1)(x-4)}\cdot\frac{x+1}{x+1}-\frac{x}{(x+1)(x-4)}\cdot\frac{x-1}{x-1}$ $\frac{x^2+3x+2}{(x-1)(x-4)(x+1)}-\frac{x^2-x}{(x+1)(x-4)(x-1)}$	$\frac{(x-4)(x-1)(x+1)}{(x-1)(x-4)}=x+1$ Multiply the first expression by $\frac{x+1}{x+1}$. $\frac{(x-4)(x-1)(x+1)}{(x+1)(x-4)}=x-1$ Multiply the second expression by $\frac{x-1}{x-1}$.
$\frac{x^2+3x+2-(x^2-x)}{(x-1)(x-4)(x+1)}$	Subtract the numerators.
$\frac{4x+2}{(x-1)(x-4)(x+1)}$	Simplify.
$\frac{x+2}{x^2-5x+4}-\frac{x}{x^2-3x-4}=\frac{4x+2}{(x-1)(x-4)(x+1)}$	

Practice Exercises E

Perform the indicated operation, if possible, simplify. Determine if your answer is a rational expression.

1. $\frac{27}{11x}-\frac{5}{11x}$

2. $\frac{14x}{2x-3}-\frac{21}{2x-3}$

3. $\frac{2x}{x+1}+\frac{2}{x+1}$

4. $\frac{3x}{x^2+6x+9}+\frac{9}{x^2+6x+9}$

5. $\frac{5x}{25x^2-49}-\frac{7}{25x^2-49}$

6. $\frac{3}{x+2}-\frac{2}{3x+6}$

7. $\frac{6}{x+1}-\frac{x}{2x+2}$

8. $\frac{5}{4x-2}+\frac{1}{10x-5}$

9. $\frac{7}{6x-30}+\frac{2}{3x-15}$

10. $\frac{2}{5x}-\frac{3}{20x^2}$

11. $\frac{3}{x^2-3x+2}-\frac{3}{x-2}$

12. $\frac{x-3}{2x-1}+\frac{x+5}{2x^2+9x-5}$

13. $\frac{1}{x+4}-\frac{3}{x^2+11x+28}$

14. $\frac{2}{3x+1}-\frac{5}{x-4}$

15. $\frac{x}{x-7}+\frac{7}{x+7}$

16. $\frac{2x}{x+5}-\frac{x}{x+8}$

17. $\frac{-6}{x-3}+\frac{5}{x-2}$

18. $\frac{4}{x^2-x}+\frac{6}{x^2-4x+3}$

19. $\frac{3}{x^2-4}+\frac{2}{x^2+5x+6}$

20. $\frac{3}{2x^2-14x}+\frac{2}{x^2-8x+7}$

21. $\frac{4}{x^2-25}-\frac{4}{x^2+10x+25}$

Simplify.

22. $\frac{4}{x+3}-\frac{10}{x-3}+\frac{x+2}{x^2-9}$

23. $\frac{8}{2x+3}+\frac{4}{x-2}-\frac{3x}{2x^2-x-6}$

24. $\frac{5}{2x}+\frac{4}{3y}+\frac{11}{6xy}$

25. $\frac{2}{x}-\frac{3}{x-1}-\frac{1}{x+2}$

YOU DECIDE

Are rational expressions closed under addition and subtraction? Justify your conclusion using the method of your choice.

Unit 2 Cluster 8 (A.REI.2): One Variable Rational and Radical Equations

Cluster 8: Understand solving equations as a process of reasoning and explain the reasoning.

2.8 Solve simple rational equations in one variable and give examples of how extraneous solutions may arise.

2.8 Solve simple radical equations in one variable and give examples of how extraneous solutions may arise.

VOCABULARY
A **rational equation** is an equation that contains one or more rational expressions (i.e., $\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{x^2-4}$ is a rational equation).
An **extraneous solution** is a solution of an equation that has been transformed or derived from the original equation but it is not a solution of the original equation. When working with rational functions you must check the solution in the original equation.

Finding Restrictions on Rational Equations

The denominator of a rational expression cannot be zero. When solving a rational equation, any values that would make any denominator zero must be excluded as possible answers. These are referred to as restrictions.

Example 1:
Find the restrictions for each rational equation.

a. $\frac{2x+1}{x+5}=1$

b. $\frac{-2}{x+8}=\frac{4x+3}{2x^2+15x-8}$

a. $\frac{2x+1}{x+5}=1$	
$x+5=0$ $x=-5$	Set the denominator equal to zero and solve for *x*.
The possible answers cannot include $x=-5$.	

b. $\frac{-2}{x+8}=\frac{4x+3}{2x^2+15x-8}$	
$\frac{-2}{x+8}=\frac{4x+2}{(2x-1)(x+8)}$	Factor the denominator of the expression on the right side.

$2x-1=0$ $2x=1$ $x=\frac{1}{2}$ $x+8=0$ $x=-8$	Set each unique factor equal to zero and solve for x. Notice that the factor $x+8$ is repeated.
The possible answers cannot include $x=-8$ and $x=\frac{1}{2}$.	

Practice Exercises A

Find the restrictions for each rational equation.

1. $\frac{3x+4}{x+9}=4$
2. $\frac{3}{2x+3}=\frac{1}{x-3}$
3. $\frac{2x}{x+6}=\frac{x}{x-1}$
4. $\frac{x+1}{x^2+3x-40}=\frac{1}{x-5}$
5. $\frac{2x-1}{x^2}=\frac{1}{x}$
6. $\frac{x^2-8x-9}{x^2+2}=1$
7. $\frac{x+5}{2x^2-2x}=\frac{2}{x}$
8. $\frac{x^2-9}{2x^2-21x+54}=\frac{1}{2}$
9. $\frac{x^2-5x-24}{3x^2-28x-20}=\frac{1}{3}$

Solving Rational Equations

To solve a rational equation:

- determine any values that would make the denominator zero
- find a common denominator by finding the least common multiple of the denominators
- multiply all of the terms on both sides of the equation by the common denominator to eliminate the fractions
- simplify and solve the resulting equation
- compare your answer with the restrictions to ensure that it is valid

Example 2:

Solve $\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$.

$\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$	
$x=0$	Determine any numbers that will make the denominator zero. The answer cannot be $x=0$.
$\frac{2}{3}\cdot\frac{6x}{1}-\frac{1}{x}\cdot\frac{6x}{1}=\frac{5}{6}\cdot\frac{6x}{1}$ $\frac{12x}{3}-\frac{6x}{x}=\frac{30x}{6}$	The common denominator is $6x$. Multiply each term by $6x$ and simplify.
$4x-6=5x$ $-6=x$	Simplify and solve the equation.

Example 3:

Solve $\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{x^2-4}$.

$\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{x^2-4}$	
$\frac{3}{x+2}+\frac{1}{x-2}=\frac{x}{(x+2)(x-2)}$	Factor the denominators.
$x+2=0$ $\quad$ $x-2=0$ $x=-2$ $\quad$ $x=2$	Find any values that will make the denominators zero. The answer cannot be $x=-2$ or $x=2$.
The common denominator is $(x+2)(x-2)$. Multiply each term by $(x+2)(x-2)$ and simplify. $\frac{3}{\cancel{x+2}}\cdot\frac{\cancel{(x+2)}(x-2)}{1}+\frac{1}{\cancel{x-2}}\cdot\frac{(x+2)\cancel{(x-2)}}{1}=\frac{x}{\cancel{(x-2)}\cancel{(x+2)}}\cdot\frac{\cancel{(x+2)}\cancel{(x-2)}}{1}$	
$3(x-2)+1(x+2)=x$ $3x-6+x+2=x$ $4x-4=x$ $-4=-3x$ $\frac{4}{3}=x$	Simplify and solve the new equation. Compare the answer against the restrictions to make sure that it is valid.

Example 4:

$$\frac{2}{x-1}+\frac{2}{x+2}=1$$

$\frac{2}{x-1}+\frac{2}{x+2}=1$	
$x-1=0$ $\quad$ $x+2=0$ $x=1$ $\quad$ $x=-2$	Find any values that will make the denominators zero. The answer cannot be $x=1$ or $x=-2$.
The common denominator is $(x-1)(x+2)$. Multiply each term by $(x-1)(x+2)$ and simplify. $\frac{2}{\cancel{x-1}}\cdot\frac{\cancel{(x-1)}(x+2)}{1}+\frac{2}{\cancel{x+2}}\cdot\frac{(x-1)\cancel{(x+2)}}{1}=1\cdot\frac{(x-1)(x+2)}{1}$	
$2(x+2)+2(x-1)=(x-1)(x+2)$ $2x+4+2x-2=x^2+x-2$ $4x+2=x^2+x-2$ $0=x^2-3x-4$ $0=(x-4)(x+1)$ $0=x-4$ $\quad$ $0=x+1$ $4=x$ $\quad$ $-1=x$	Simplify and solve the new equation. Compare the answer against the restrictions to make sure that it is valid.

Example 5:

Solve $\frac{5x}{x+1}=4-\frac{5}{x+1}$.

$\frac{5x}{x+1}=4-\frac{5}{x+1}$	
$x+1=0$ $x=-1$	Find any values that will make the denominators zero. The answer cannot be $x=-1$.
$\frac{5x}{\cancel{x+1}}\cdot\frac{\cancel{x+1}}{1}=4\cdot(x+1)-\frac{5}{\cancel{x+1}}\cdot\frac{\cancel{x+1}}{1}$ $5x=4(x+1)-5$	The common denominator is $(x+1)$. Multiply each term by $(x+1)$ and simplify.
$5x=4x+4-5$ $5x=4x-1$ $x=-1$	Simplify and solve the equation.
The mathematical answer is $x=-1$, but this value will make the denominator zero, therefore, $x=-1$ is an extraneous solution and there is no solution to this equation.	

Practice Exercises B

Solve each equation.

1. $\dfrac{11}{3x}-\dfrac{1}{3}=\dfrac{-4}{x^2}$

2. $\dfrac{3}{2x}-\dfrac{5}{3x}=2$

3. $\dfrac{1}{4x}-\dfrac{3}{4}=\dfrac{7}{x}$

4. $\dfrac{2}{4}-\dfrac{3}{2x}=\dfrac{1}{x}$

5. $\dfrac{x}{x-7}=\dfrac{49}{x^2-7x}$

6. $\dfrac{3}{x-1}+\dfrac{3}{10}=\dfrac{5}{2x-2}$

7. $\dfrac{x}{x+1}+\dfrac{5}{x}=\dfrac{1}{x^2+x}$

8. $\dfrac{4}{x^2+4x-5}+\dfrac{7}{x+5}=\dfrac{5}{x-1}$

9. $\dfrac{4}{x^2-9}-\dfrac{2}{x+3}=\dfrac{3}{2x-6}$

10. $\dfrac{x}{x-2}+\dfrac{3}{x-1}=1$

11. $\dfrac{2x}{x+3}=1-\dfrac{6}{x+3}$

12. $\dfrac{2}{x+3}+\dfrac{3}{x}=\dfrac{10}{x^2+3x}$

13. $\dfrac{8}{x^2+8x+12}=\dfrac{4}{x+6}+\dfrac{4}{x+2}$

14. $\dfrac{4}{x-2}-\dfrac{2}{x^2-4}=\dfrac{6}{x+2}$

15. $\dfrac{5}{x^2-7x+12}-\dfrac{2}{3-x}=\dfrac{5}{x-4}$

16. $\dfrac{6}{x+1}=\dfrac{x}{x-1}$

17. $\dfrac{4}{x-7}=\dfrac{-2x}{x+3}$

18. $\dfrac{1}{x+6}=\dfrac{36}{x^2+6x}$

19. $\dfrac{5x}{x-2}=7+\dfrac{10}{x-2}$

20. $\dfrac{3}{x-2}=\dfrac{5}{x+4}$

21. $x-\dfrac{12}{x}=4$

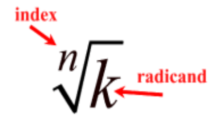
22. $\dfrac{2}{x-4}+\dfrac{1}{x}=\dfrac{x}{4-x}$

23. $\dfrac{5}{4x}=\dfrac{7}{5x-2}$

24. $\dfrac{x+3}{x+2}-\dfrac{x}{x^2-4}=\dfrac{x}{x-2}$

VOCABULARY

A **radical equation** is an equation that has a variable in a radicand or a variable with a rational exponent (i.e, $\sqrt{2x+3}=4$ or $(4x-1)^{1/3}=1$). The **radicand** is the expression under the radical sign. The **index** is the small number outside of the radical sign.



Solving Radical Equations

To solve a radical equation:

- isolate the radical on one side of the equation
- raise each side to the power of the index
- simplify
- check solutions in the original equation to eliminate any extraneous solutions

Example 6:

Solve $4+\sqrt{3x+10}=9$.

$4+\sqrt{3x+10}=9$	
$\sqrt{3x+10}=5$	Isolate the radical term by subtracting 4 from each side.
$\left(\sqrt{3x+10}\right)^2=5^2$ $3x+10=25$	Square each side of the equation.
$3x=15$ $x=5$	Solve for *x*.
$4+\sqrt{3\cdot5+10}\overset{?}{=}9$ $4+\sqrt{15+10}\overset{?}{=}9$ $4+\sqrt{25}\overset{?}{=}9$ $4+5\overset{?}{=}9$ $9=9$	Check the solution in the original equation.

Example 7:

Solve $-1+\sqrt[3]{2x-5}=2$.

$-1+\sqrt[3]{2x-5}=2$	
$\sqrt[3]{2x-5}=3$	Isolate the radical term by adding 1 to each side.
$\left(\sqrt[3]{2x-5}\right)^3=3$ $2x-5=27$	Cube each side of the equation.
$2x=32$ $x=16$	Solve for *x*.
$-1+\sqrt[3]{2\cdot 16-5}\overset{?}{=}2$ $-1+\sqrt[3]{32-5}\overset{?}{=}2$ $-1+\sqrt[3]{27}\overset{?}{=}2$ $-1+3\overset{?}{=}2$ $2=2$	Check the solution in the original equation.

Example 8:

Solve $\sqrt{x+9}-7=x$.

$\sqrt{x+9}-7=x$	
$\sqrt{x+9}=x+7$	Isolate the radical term by adding 7 to each side.
$\left(\sqrt{x+9}\right)^2=(x+7)^2$ $x+9=x^2+14x+49$	Square each side of the equation. Remember that $(x+7)^2=(x+7)(x+7)$.
$0=x^2+13x+40$ $0=(x+5)(x+8)$ $x+5=0 \qquad x+8=0$ $x=-5 \qquad x=-8$	Solve for *x*.

$\sqrt{-5+9}-7\stackrel{?}{=}-5$ $\sqrt{4}-7\stackrel{?}{=}-5$ $2-7\stackrel{?}{=}-5$ $-5=-5$	$\sqrt{-8+9}-7\stackrel{?}{=}-8$ $\sqrt{1}-7\stackrel{?}{=}-8$ $1-7\stackrel{?}{=}-8$ $-6\neq-8$	Check the solutions in the original equation.
The only solution is $x=-5$ because $x=-8$ does not work in the original equation so it is an extraneous solution.		

Example 9:

Solve $\sqrt{3x+1}-\sqrt{x+1}=2$.

$\sqrt{3x+1}-\sqrt{x+1}=2$		
$\sqrt{3x+1}=2+\sqrt{x+1}$		Isolate one of the radical terms.
$\left(\sqrt{3x+1}\right)^2=\left(2+\sqrt{x+1}\right)^2$ $3x+1=4+4\sqrt{x+1}+x+1$ $3x+1=4\sqrt{x+1}+x+5$		Square each side of the equation. Remember that $\left(2+\sqrt{x+1}\right)^2=\left(2+\sqrt{x+1}\right)\left(2+\sqrt{x+1}\right)$.
$2x-4=4\sqrt{x+1}$		Isolate the radical term.
$(2x-4)^2=\left(4\sqrt{x+1}\right)^2$ $4x^2-16x+16=16(x+1)$		Square each side of the equation. Remember that $(2x-4)^2=(2x-4)(2x-4)$.
$4x^2-16x+16=16x+16$ $4x^2-32x=0$ $4x(x-8)=0$ $4x=0$ $\quad$ $x-8=0$ $x=0$ $\quad$ $x=8$		Solve for *x*.
$\sqrt{3\cdot 0+1}-\sqrt{0+1}\stackrel{?}{=}2$ $\sqrt{1}-\sqrt{1}\stackrel{?}{=}2$ $0\neq 2$	$\sqrt{3\cdot 8+1}-\sqrt{8+1}\stackrel{?}{=}2$ $\sqrt{25}-\sqrt{9}\stackrel{?}{=}2$ $5-3\stackrel{?}{=}2$ $2=2$	Check the solutions in the original equation.
The only solution is $x=8$ because $x=0$ does not work in the original equation so it is an extraneous solution.		

Practice Exercises C

Solve each radical equation.

1. $\sqrt{x-2}+5=8$	2. $\sqrt[3]{x-2}+1=4$	3. $2\sqrt{x+4}-5=-3$
4. $\sqrt[4]{x-10}+5=8$	5. $\sqrt{2x-1}-3=2$	6. $\sqrt[5]{x-1}+4=5$
7. $3\sqrt{x}-4=11$	8. $\sqrt[5]{x+3}+7=5$	9. $\sqrt{3x+4}+6=13$
10. $\sqrt[4]{x+5}-7=-5$	11. $\sqrt{x+3}-2=4$	12. $-5\sqrt[3]{x}-9=11$
13. $-\sqrt{x-4}+3=-1$	14. $\sqrt[4]{2x+3}-2=1$	15. $\sqrt{x-2}+4=2$
16. $\sqrt{3x+7}+1=x$	17. $\sqrt[3]{3x+4}+1=2$	18. $\sqrt[3]{6x+9}+8=5$
19. $2\sqrt[3]{x}+6=-4$	20. $\sqrt{11x+3}=2x$	21. $3\sqrt[5]{x+6}-7=-4$
22. $\sqrt[3]{21x+55}-2=8$	23. $\sqrt{x+7}=x-5$	24. $\sqrt[3]{x+3}-8=-6$
25. $\sqrt{4x-3}=2+\sqrt{2x-5}$	26. $\sqrt{2x+6}=2+\sqrt{x-1}$	27. $\sqrt{3-x}+\sqrt{x+2}=3$

HONORS

Recall that when a rational number is multiplied by its reciprocal the product is 1 (i.e., $\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ab}=1$). To solve radical equations of the form $x^{\frac{a}{b}}=k$, raise each side of the equation to the power of the reciprocal $\frac{b}{a}$.

Example 10:

Solve $2(x+1)^{2/3}+5=13$.

$2(x+1)^{2/3}+5=13$	
$2(x+1)^{2/3}=8$ $(x+1)^{2/3}=4$	Isolate the radical term.

$\left((x+1)^{2/3}\right)^{3/2}=4^{3/2}$ $x+1=\left(\sqrt{4}\right)^3$ $x+1=(\pm 2)^3$ $x+1=2^3$ $x+1=8$ $x=7$	$x+1=(-2)^3$ $x+1=-8$ $x=-9$	The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Raise each side of the equation to the $\frac{3}{2}$ power. Rewrite the using the properties of exponents. Remember that the square root of a number has a positive and a negative solution.
$2(7+1)^{2/3}+5\overset{?}{=}13$ $2(8)^{2/3}+5\overset{?}{=}13$ $2(4)+5\overset{?}{=}13$ $8+5\overset{?}{=}13$ $13=13$	$2(-9+1)^{2/3}+5\overset{?}{=}13$ $2(-8)^{2/3}+5\overset{?}{=}13$ $2(4)+5\overset{?}{=}13$ $8+5\overset{?}{=}13$ $13=13$	Check the solutions in the original equation.
Both $x=7$ and $x=-9$ are solutions to the radical equation.		

Example 11:

Solve $4\sqrt[5]{(x+12)^3}-6=26$.

$4\sqrt[5]{(x+12)^3}-6=26$	
$4\sqrt[5]{(x+12)^3}=32$ $\sqrt[5]{(x+12)^3}=8$	Isolate the radical term.
$(x+12)^{3/5}=8$	Rewrite the expression in rational exponent form.
$\left((x+12)^{3/5}\right)^{5/3}=8^{5/3}$ $x+12=\left(\sqrt[3]{8}\right)^5$ $x+12=(2)^5$ $x+12=32$ $x=20$	The reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$. Raise each side of the equation to the $\frac{5}{3}$ power. Rewrite the using the properties of exponents. Remember that the cube root has only one real answer.

$4\sqrt[5]{(20+12)^3}-6\stackrel{?}{=}26$ $4\sqrt[5]{(32)^3}-6\stackrel{?}{=}26$ $4\sqrt[5]{32,768}-6\stackrel{?}{=}26$ $4\cdot 8-6\stackrel{?}{=}26$ $32-6\stackrel{?}{=}26$ $26=26$	Check the solution in the original equation.
$x=20$ is a solution to the radical equation.	

Practice Exercises D

Solve each radical equation.

1. $(x+7)^{2/3}+6=10$
2. $2\sqrt[3]{(x+15)^2}-5=13$
3. $5(x-4)^{3/4}-25=15$
4. $(x+9)^{3/4}-15=12$
5. $(x+8)^{3/2}-6=21$
6. $-2\sqrt{(3-x)^3}+5=-11$
7. $\sqrt[3]{(x+2)^5}-10=22$
8. $\frac{1}{3}(23x+13)^{3/5}+2=11$
9. $\frac{1}{2}(x+20)^{4/3}-5=3$

Graphing Functions

Unit 4 Clusters 3 and 5 (F.IF.7b,e and F.BF.3): Transformations

Cluster 3: Analyze functions using different representations

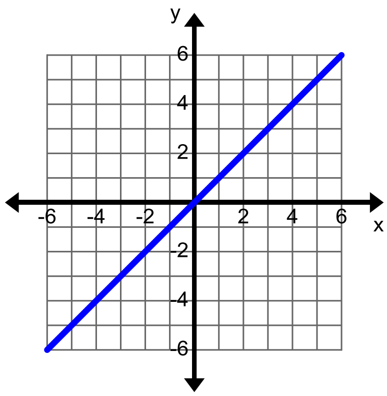
4.3 Graph basic functions i.e., square root, cube root, piecewise-defined functions, step functions, absolute value functions, exponential, logarithmic, trigonometric functions, and polynomial functions (including linear and quadratic functions) with and without technology.

Cluster 5: Build new functions from existing functions

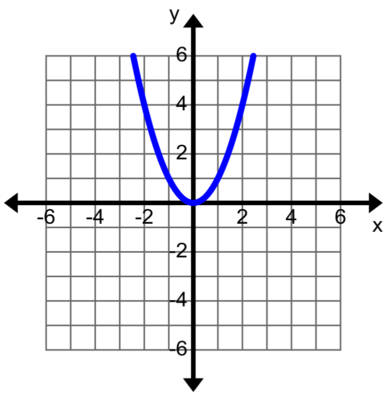
4.5 Identify the effect on the graph by replacing $f(x)$ with $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs, with and without technology. Include recognizing even and odd functions graphically and algebraically.

VOCABULARY
There are several types of functions (linear, exponential, quadratic, absolute value, etc.). Each of these could be considered a family with unique characteristics that are shared among the members. The **parent function** is the basic function that is used to create more complicated functions.

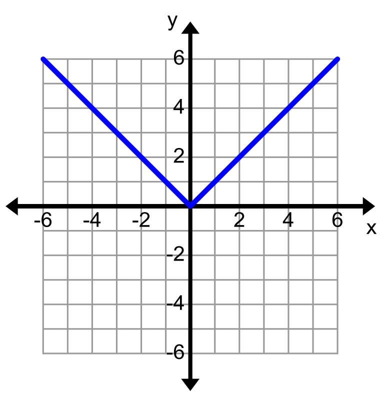
Linear Function

Parent Function $f(x)=x$	Key Features
	**Domain:** $(-\infty,\infty)$ **Range:** $(-\infty,\infty)$ **Intercepts**: x-intercept $(0,0)$, y-intercept $(0,0)$ **Intervals of Increasing/Decreasing:** increasing $(-\infty,\infty)$ **Intervals where Positive/Negative:** positive $(0,\infty)$, negative $(-\infty,0)$ **Relative maximums/minimums**: none **Symmetries:** odd **End Behavior**: right end behavior $\lim_{x\to\infty} x=\infty$; left end behavior $\lim_{x\to-\infty} x=-\infty$

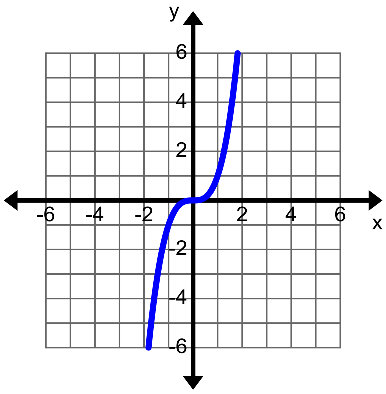
Quadratic Function

Parent Function $f(x)=x^2$ 	Key Features **Domain:** $(-\infty,\infty)$ **Range:** $[0,\infty)$ **Intercepts**: x-intercept $(0,0)$, y-intercept $(0,0)$ **Intervals of Increasing/Decreasing:** increasing $(0,\infty)$, decreasing $(-\infty,0)$ **Intervals where Positive/Negative:** positive $(-\infty,0)\cup(0,\infty)$ **Relative maximums/minimums**: minimum at $(0,0)$ **Symmetries:** even **End Behavior**: right end behavior $\lim\limits_{x\to\infty} x^2=\infty$; left end behavior $\lim\limits_{x\to-\infty} x^2=\infty$

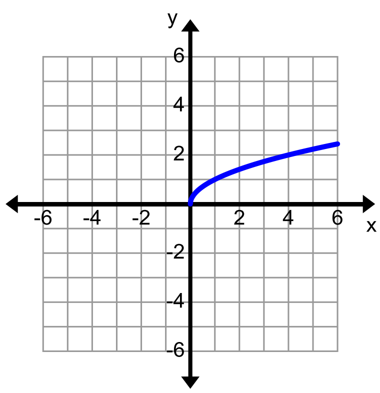
Absolute Value Function

Parent Function $f(x)=\lvert x\rvert$ 	Key Features **Domain:** $(-\infty,\infty)$ **Range:** $[0,\infty)$ **Intercepts**: x-intercept $(0,0)$, y-intercept $(0,0)$ **Intervals of Increasing/Decreasing:** increasing $(0,\infty)$, decreasing $(-\infty,0)$ **Intervals where Positive/Negative:** positive $(-\infty,0)\cup(0,\infty)$ **Relative maximums/minimums**: minimum at $(0,0)$ **Symmetries:** even **End Behavior**: right end behavior $\lim\limits_{x\to\infty}\lvert x\rvert=\infty$; left end behavior $\lim\limits_{x\to-\infty}\lvert x\rvert=\infty$

Cubic Function

Parent Function $f\left(x\right)=x^3$  	Key Features **Domain:** $\left(-\infty,\infty\right)$ **Range:** $\left(-\infty,\infty\right)$ **Intercepts**: x-intercept $\left(0,0\right)$, y-intercept $\left(0,0\right)$ **Intervals of Increasing/Decreasing:** increasing $\left(-\infty,\infty\right)$ **Intervals where Positive/Negative:** positive $\left(0,\infty\right)$, negative $\left(-\infty,0\right)$ **Relative maximums/minimums**: none **Symmetries:** odd **End Behavior**: right end behavior $\lim\limits_{x\to\infty} x^3=\infty$; left end behavior $\lim\limits_{x\to-\infty} x^3=-\infty$

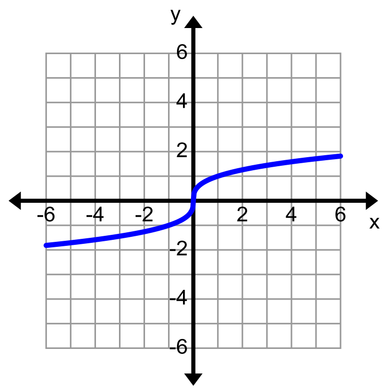
Square Root Function

Parent Function $f\left(x\right)=\sqrt{x}=x^{1/2}$  	Key Features **Domain:** $\left[0,\infty\right)$ **Range:** $\left[0,\infty\right)$ **Intercepts**: x-intercept $\left(0,0\right)$, y-intercept $\left(0,0\right)$ **Intervals of Increasing/Decreasing:** increasing $\left(0,\infty\right)$ **Intervals where Positive/Negative:** $\left(0,\infty\right)$ **Relative maximums/minimums**: minimum at $\left(0,0\right)$ **Symmetries:** none **End Behavior**: right end behavior $\lim\limits_{x\to\infty} \sqrt{x}=\infty$; left end behavior $\lim\limits_{x\to0^+} \sqrt{x}=0$

Cube Root Function

Parent Function

$f(x)=\sqrt[3]{x}=x^{1/3}$



Key Features

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

Intercepts: x-intercept $(0,0)$, y-intercept $(0,0)$

Intervals of Increasing/Decreasing: increasing $(-\infty,\infty)$

Intervals where Positive/Negative: positive $(0,\infty)$, negative $(-\infty,0)$

Relative maximums/minimums: none

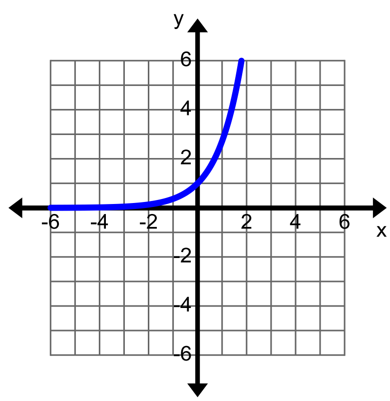
Symmetries: odd

End Behavior: right end behavior $\lim_{x\to\infty}\sqrt[3]{x}=\infty$; left end behavior $\lim_{x\to-\infty}\sqrt[3]{x}=-\infty$

Exponential Function

Parent Function

$f(x)=a^x$ or $f(x)=e^x$



Key Features

Domain: $(-\infty,\infty)$

Range: $(0,\infty)$

Intercepts: y-intercept $(0,1)$

Intervals of Increasing/Decreasing: increasing $(-\infty,\infty)$

Intervals where Positive/Negative: positive $(-\infty,\infty)$

Relative maximums/minimums: none

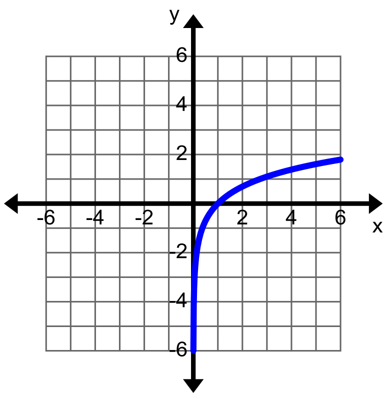
Symmetries: none

End Behavior: right end behavior $\lim_{x\to\infty}e^x=\infty$; left end behavior $\lim_{x\to-\infty}e^x=0$

Logarithmic Function

Parent Function

$f(x) = \log_b x$ or $f(x) = \ln x$



Key Features

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Intercepts: x-intercept $(1, 0)$

Intervals of Increasing/Decreasing: increasing $(0, \infty)$

Intervals where Positive/Negative: positive $(1, \infty)$, negative $(0, 1)$

Relative maximums/minimums: none

Symmetries: none

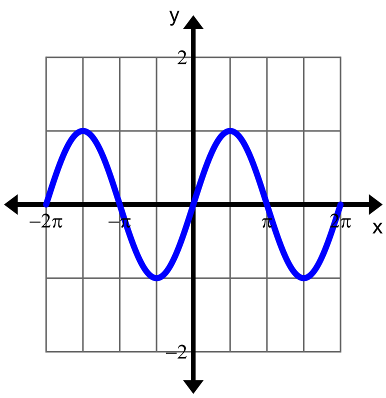
End Behavior: right end behavior $\lim_{x \to \infty} \ln x = \infty$; left end behavior $\lim_{x \to 0^+} \ln x = -\infty$

Note: There is a vertical asymptote at $x = 0$.

Sine Function

Parent Function

$f(x) = \sin x$



Key Features

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Intercepts: x-intercepts $(\pm k\pi, 0)$, y-intercept $(0, 0)$

Intervals of Increasing/Decreasing: alternating increasing and decreasing in periodic waves

Intervals where Positive/Negative: alternating positive and negative in periodic waves

Relative maximums/minimums: absolute maximum of 1 and absolute minimum of -1

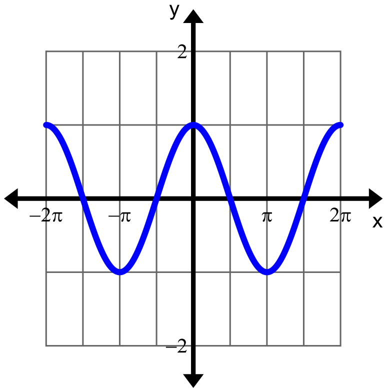
Symmetries: odd

End Behavior: no end behavior because the values oscillate between -1 and 1 and approach no limit

Cosine Function

Parent Function

$f(x) = \cos x$



Key Features

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Intercepts: x-intercepts $\left(\pm\frac{k\pi}{2}, 0\right)$ where k is odd, y-intercept $(0,1)$

Intervals of Increasing/Decreasing: alternating increasing and decreasing in periodic waves

Intervals where Positive/Negative: alternating positive and negative in periodic waves

Relative maximums/minimums: absolute maximum of 1 and absolute minimum of -1

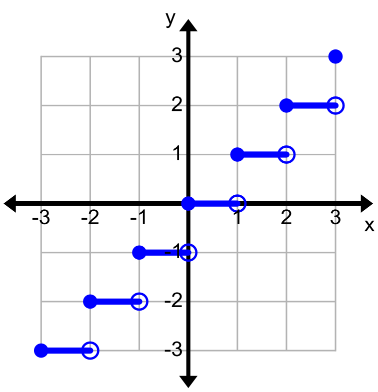
Symmetries: even

End Behavior: no end behavior because the values oscillate between -1 and 1 and approach no limit

Step Functions are piecewise-defined functions made up of constant functions. It is called a step function because the graph resembles a staircase.

Step Function

$f(x) = \text{int } x$



Key Features

Domain: $(-\infty, \infty)$

Range: $\{y \mid y \text{ is an integer}\}$

Intercepts: x-intercept $x = [0,1)$ and $y = 0$, y-intercept $(0,0)$

Intervals of Increasing/Decreasing: neither increasing nor decreasing

Intervals where Positive/Negative: positive $(1, \infty)$, negative $(-\infty, 0)$

Relative maximums/minimums: none

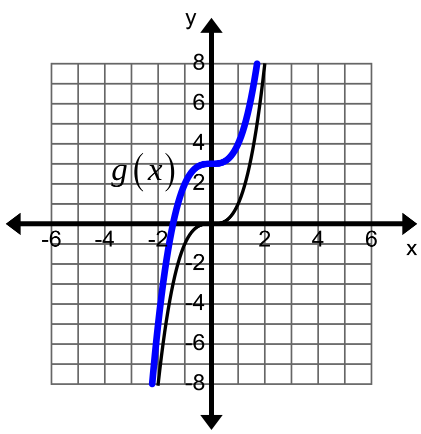
Symmetries: none

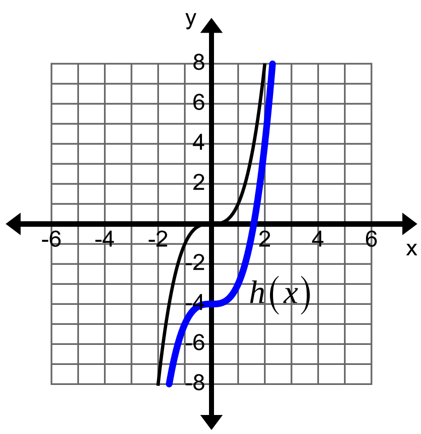
End Behavior: right end behavior $\lim_{x\to\infty} \text{int } x = \infty$; left end behavior $\lim_{x\to-\infty} \text{int } x = -\infty$

Example 1: Vertical Shift

Given $f(x)=x^3$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(x)+3$ b. $h(x)=f(x)-2$

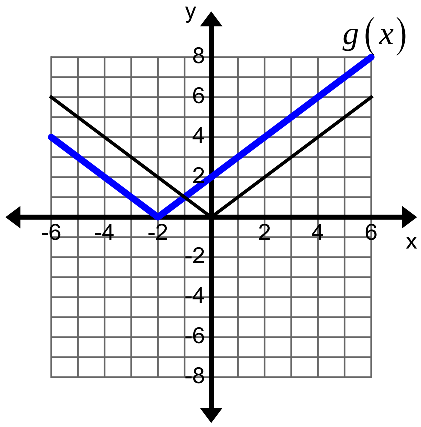
a. $g(x)=f(x)+3$ 	$k=3$ so the graph is shifted up 3 units. The transformed function is neither odd nor even.

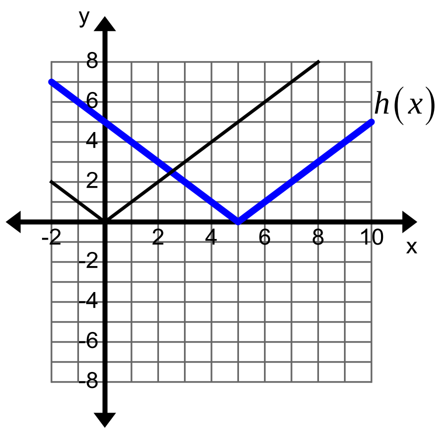
b. $h(x)=f(x)-4$ 	$k=-4$ so the graph is shifted down 4 units. The transformed function is neither odd nor even.

Example 2: Horizontal Shift

Given $f(x)=|x|$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(x+2)$ b. $h(x)=f(x-5)$

a. $g(x)=f(x+2)$ 	$k=2$ so the graph is shifted left 2 units. The transformed function is neither odd nor even.

b. $h(x)=f(x-5)$ 	$k=-5$ so the graph is shifted right 5 units. The transformed function is neither odd nor even.

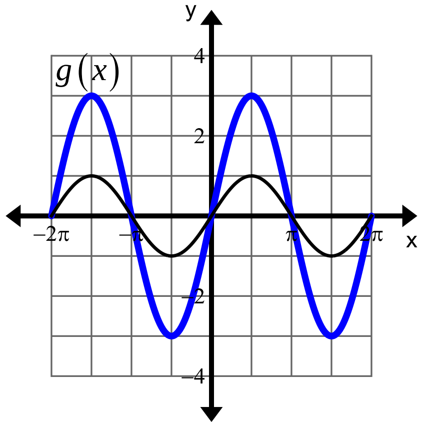
Example 3: Vertical Stretch

Given $f(x) = \cos x$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x) = 3f(x)$

b. $h(x) = \frac{1}{3} f(x)$

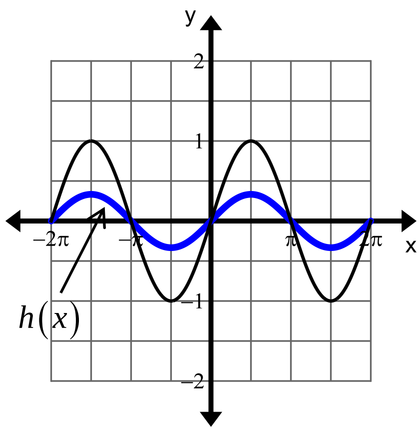
a. $g(x) = 3f(x)$



$k = 3$ so the graph is stretched vertically by a factor of 3.

The transformed function is odd.

b. $h(x) = \frac{1}{3} f(x)$



$k = \frac{1}{3}$ so the graph is stretched vertically by a factor of $\frac{1}{3}$.

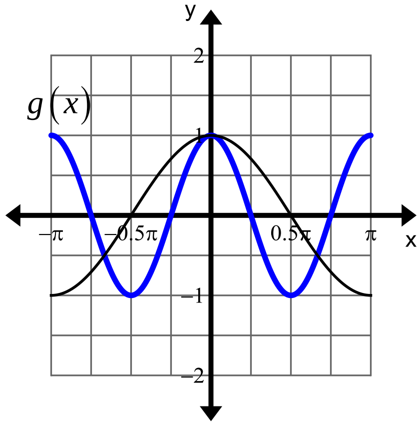
The transformed function is odd.

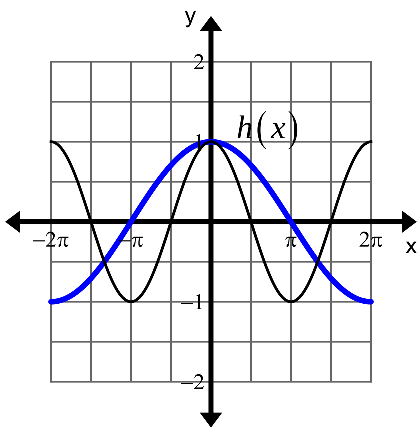
Example 4: Horizontal Stretch

Given $f(x)=\cos x$, graph each new function, without technology, and describe the effect of k on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(2x)$

b. $h(x)=f\left(\frac{1}{2}x\right)$

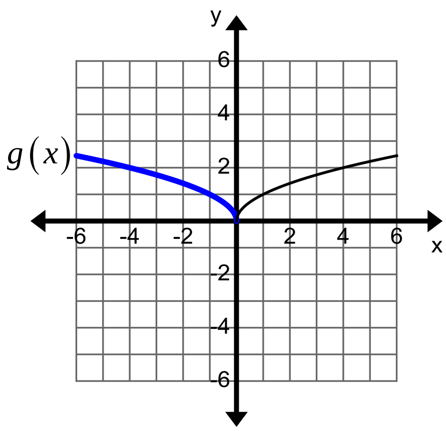
a. $g(x)=f(2x)$ 	$k=2$ so the graph is stretched horizontally by a factor of $\frac{1}{2}$. Notice that the transformed function has more cycles than the original function. The transformed function is even.

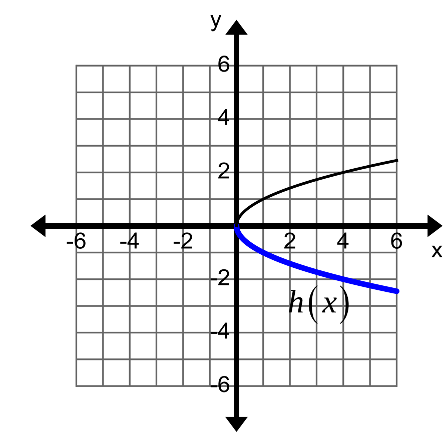
b. $h(x)=f\left(\frac{1}{2}x\right)$ 	$k=\frac{1}{2}$ so the graph is stretched horizontally by a factor of 2. Notice that the transformed function has fewer cycles than the original function. The transformed function is even.

Example 5: Reflections

Given $f(x)=\sqrt{x}$, graph each new function, without technology, and describe the effect of *k* on the original graph. Determine if the transformed function is even, odd, or neither.

a. $g(x)=f(-x)$ b. $h(x)=-f(x)$

a. $g(x)=f(-x)$ 	$k=-1$ so the graph is reflected across the *y*-axis. The transformed function is neither odd nor even.

b. $h(x)=-f(x)$ 	$k=-1$ so the graph is reflected across the *x*-axis. The transformed function is neither odd nor even.

Practice Exercises A

Given $f(x)$, graph each new function, without technology, and describe the effect of *k* on the original graph. Determine if the transformed function is even, odd, or neither.

1. $f(x) = x$
   a. $g(x) = -f(x)$
   b. $h(x) = \frac{1}{2} f(x)$
   c. $j(x) = f(x) - 5$

2. $f(x) = x^2$
   a. $g(x) = f(x+6)$
   b. $h(x) = f(x) + 4$
   c. $j(x) = 2f(x)$

3. $f(x) = |x|$
   a. $g(x) = -3f(x) + 4$
   b. $h(x) = f(x-7) + 2$
   c. $j(x) = \frac{1}{4} f(x+1)$

4. $f(x) = \sqrt{x}$
   a. $g(x) = f(-x)$
   b. $h(x) = -f(x+3)$
   c. $j(x) = 2f(x) - 3$

5. $f(x) = x^3$
   a. $g(x) = -\frac{1}{4} f(x-2)$
   b. $h(x) = f(x) + 6$
   c. $j(x) = f(x+4)$

6. $f(x) = \sqrt[3]{x}$
   a. $g(x) = f(x) - 1$
   b. $h(x) = f(x+2) - 4$
   c. $j(x) = 3f(x-2)$

7. $f(x) = e^x$
   a. $g(x) = f(-x)$
   b. $h(x) = f(x+5)$
   c. $j(x) = f(x) - 2$

8. $f(x) = \ln(x)$
   a. $g(x) = 2f(-x)$
   b. $h(x) = -f(x) + 3$
   c. $j(x) = f(x-1) - 1$

9. $f(x) = \sin x$
   a. $g(x) = f(2x)$
   b. $h(x) = -\frac{3}{2} \sin x$
   c. $j(x) = f(-x)$

10. $f(x) = \cos x$
   a. $g(x) = -2f(x)$
   b. $h(x) = f\left(\frac{1}{3}x\right)$
   c. $j(x) = f(x) + 2$

11. $f(x) = \operatorname{int} \ x$
   a. $g(x) = f(-x)$
   b. $h(x) = f\left(\frac{1}{2}x\right)$
   c. $j(x) = 2f(x)$

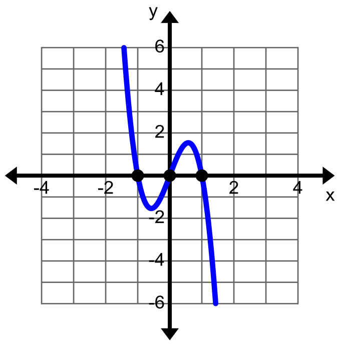
Example 6:

Determine whether each function can be obtained from the parent function, $f(x)=x^n$, using basic transformations. If so, describe the sequence of transformations.

a. $g(x)=-4x^3+4x$

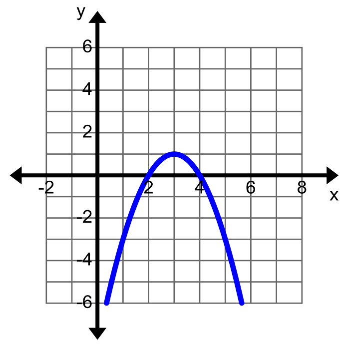
b. $h(x)=x^2-6x+8$

a. $g(x)=-4x^3+4x$



By looking that the graph of $g(x)$ you can see that it has three real roots, but the parent function $f(x)=x^3$ has only one real root. Since the parent function is increasing on its entire domain it is not possible to obtain $g(x)$ through basic transformations such as stretches, reflections, and translations.

b. $h(x)=-x^2+6x-8$



By looking at the graph of $h(x)$ you can see that the function has been reflected over the *x*-axis, translated up 1 unit and right 3 units from the parent function $f(x)=x^2$.

You could also have used the process of completing the square rewrite $h(x)$ in vertex form.

$$h(x)=-\left(x^2-6x+\_\_\_\right)-8$$

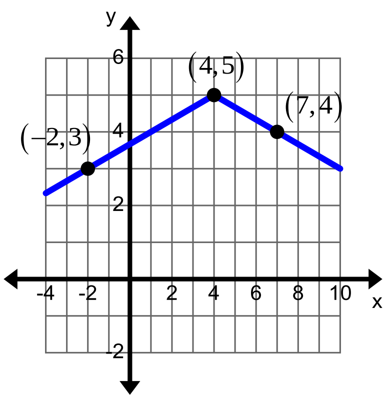
$$h(x)=-\left(x^2-6x+\left(-\frac{6}{2}\right)^2\right)-8+9$$

$$h(x)=-(x-3)^2+1$$

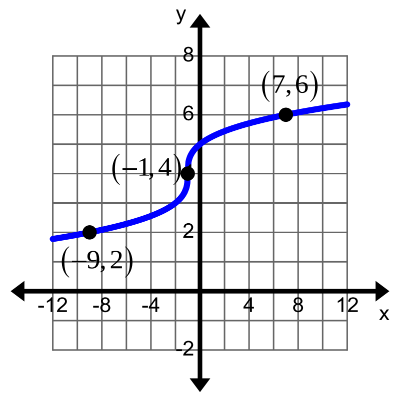
Example 7:

Determine the transformations that were used to change the given parent function to the function that is graphed.

a. $f(x) = |x|$



b. $y = \sqrt[3]{x}$



a. The transformed absolute value function opens down so it has been reflected across the *x*-axis. The first point away from the vertex is down 1 and over three so it has been vertically stretched by a factor of $\frac{1}{3}$. The vertex is at $(4,5)$ so it has been translated up 5 units and right 4 units. • Reflection over the *x*-axis • Vertical stretch by a factor of $\frac{1}{3}$ • Translation of 5 units up and 4 units right	b. The middle of the s-curve is normally at $(0,0)$ but is now at $(-1,4)$. The cube root of 8 is 2 so there should be a point at $(8,2)$, but there is a point at $(7,6)$ instead. The transformed graph has been translated up 4 units and left 1 unit. • Translation up 4 units and left 1 unit.

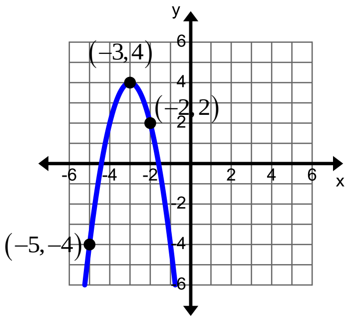
Practice Exercises B

Determine whether each function can be obtained from the parent function, $f(x)=x^n$, using basic transformations. If so, describe the sequence of transformations.

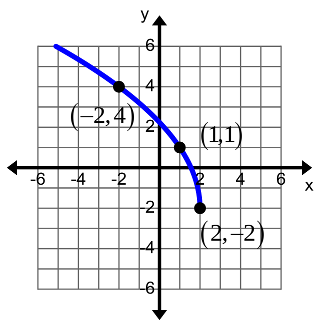
1. $g(x)=x^2-4$
2. $g(x)=x^2-2x-15$
3. $g(x)=4x^2+4x-3$
4. $g(x)=x^3-6x^2+12x-8$
5. $g(x)=-2x^3$
6. $g(x)=-3x^3+12x$
7. $g(x)=(x+3)^4-5$
8. $g(x)=x^4-8x^2+7$
9. $g(x)=2x^4-x^2$
10. $g(x)=-3x^5+2$
11. $g(x)=(x+1)^5$
12. $g(x)=(x+5)^5+7$

Determine the transformations that were used to change the given parent function to the function that is graphed.

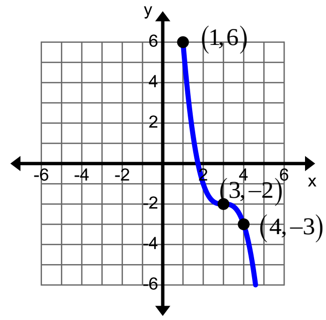
13. $f(x)=x^2$



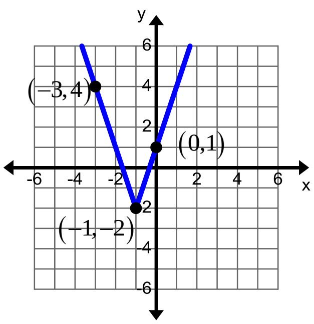
14. $f(x)=\sqrt{x}$



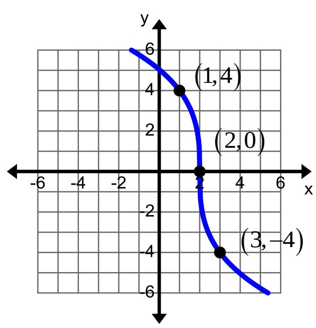
15. $f(x)=x^3$



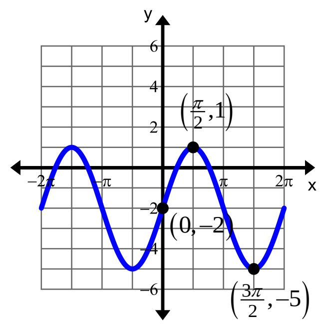
16. $f(x)=|x|$



17. $f(x)=\sqrt[3]{x}$



18. $f(x)=\sin(x)$



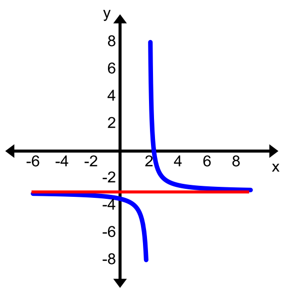
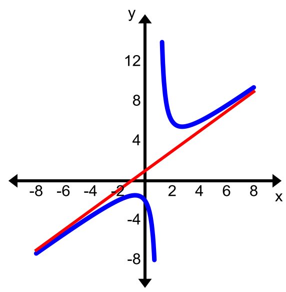
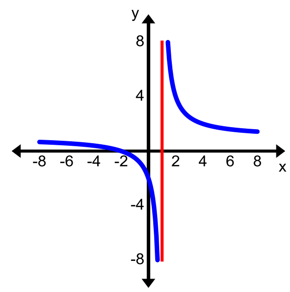
Unit 4 Clusters 3 HONORS (F.IF.7d): Graphing Rational Functions

Cluster 3: Analyze functions using different representations

4.3 Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior

VOCABULARY

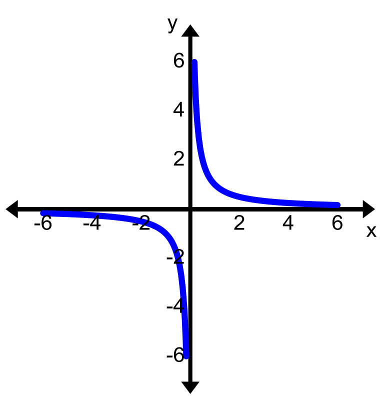
An **asymptote** is a line that a graph approaches, as *x* increases or decreases, but does not intersect. **Horizontal** and **oblique** asymptotes occur with rational functions and model end behavior. Vertical asymptotes occur when the denominator is equal to zero, but the numerator is not zero. The graph may cross a horizontal or an oblique asymptote near the origin.

Horizontal Asymptote	Oblique Asymptote	Vertical Asymptote
 $y=-3$	 $y=x+1$	 $x=1$

Reciprocal Function

Parent Function

$$f(x)=\frac{1}{x}$$



Key Features

Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(-\infty,0)\cup(0,\infty)$

Intercepts: none

Intervals of Increasing/Decreasing: decreasing $(-\infty,0)\cup(0,\infty)$

Intervals where Positive/Negative: positive $(0,\infty)$, negative $(-\infty,0)$

Relative maximums/minimums: none

Symmetries: odd

End Behavior: right end behavior $\lim_{x\to\infty} x=0$; left end behavior $\lim_{x\to-\infty} x=0$

Let $p(x)$ and $q(x)$ be polynomials with no common factors other than 1. The graph of the rational function $f(x)=\frac{p(x)}{q(x)}=\frac{a_m x^m + a_{m-1}x^{m-1}+...+a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1}+...+b_1 x + b_0}$ has the following characteristics:

1. The *x*-intercepts of the graph of *f* are the real zeros of numerator.
2. The graph of *f* has a vertical asymptote at each real zero of the denominator unless the numerator and the denominator share a factor.
3. The graph of *f* has at most one horizontal asymptote.
    - If the degree of $p(x)$ is less than the degree of $q(x)$, then the line $y=0$ is the horizontal asymptote.
    - If the degree of $p(x)$ is equal to the degree of $q(x)$, then the line $y=\frac{a}{b}$ is the horizontal asymptote.
    - If degree of $p(x)$ is greater than the degree of $q(x)$, then there may be an oblique asymptote which can be found using long division. The oblique asymptote will be in the form $y=mx+b$.

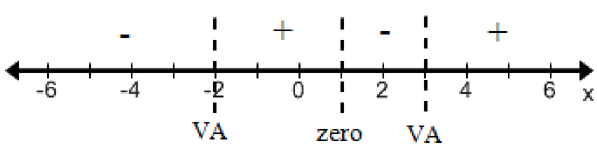
Sketching Rational Functions

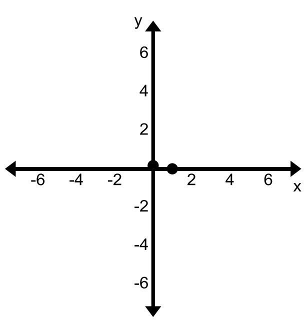
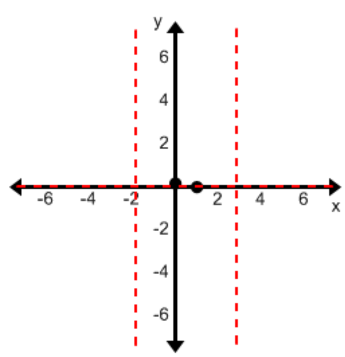
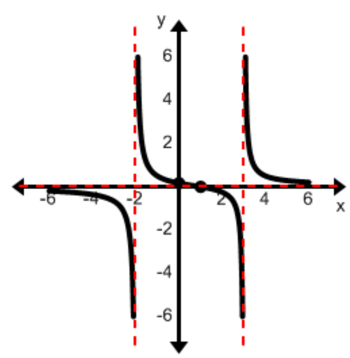
1. Factor the numerator and the denominator.
2. Find the *x*-intercept and *y*-intercepts.
3. Find the horizontal or oblique and vertical asymptotes.
4. Create a sign array to determine where the function is positive or negative.
5. Sketch the graph.
6. Identify the domain and range.

Example 1:

Graph the function $f(x)=\dfrac{x-1}{x^2-x-6}$ and identify the domain and range.

<table>
<tr><td colspan="2">$f(x)=\dfrac{x-1}{(x-3)(x+2)}$</td><td>Factor the numerator and the denominator.</td></tr>
<tr><td>$x-1=0$
$x=1$
$(1,0)$</td><td>$f(0)=\dfrac{0-1}{0^2-0-6}$
$f(0)=\dfrac{-1}{-6}$
$f(0)=\dfrac{1}{6}$
$\left(0,\dfrac{1}{6}\right)$</td><td>Find the intercepts.</td></tr>
<tr><td>$(x-3)(x+2)=0$
$x-3=0 \mid x+2=0$
$x=3 \mid x=-2$</td><td>$y=0$</td><td>Find the asymptotes.

The degree of the numerator (1) is less than the degree of the denominator (2) so the horizontal asymptote is $y=0$.</td></tr>
<tr><td colspan="2">
</td><td>The sign array determines whether the graph is positive or negative on the interval.</td></tr>
</table>

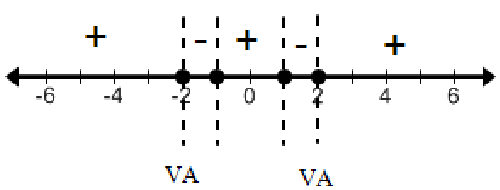
Intercepts	Asymptotes	Final Sketch
		

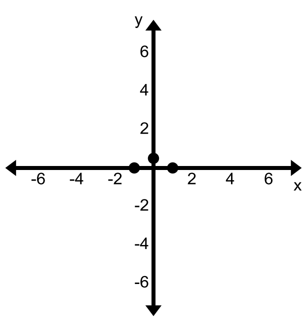
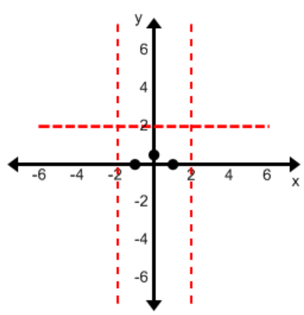
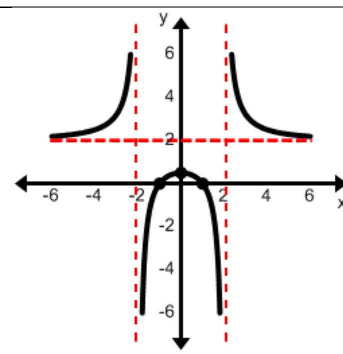
Domain: $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$ Range: $(-\infty,0)\cup(0,\infty)$	The domain includes all values except for the vertical asymptotes. The range includes all values except for the horizontal asymptote.

Example 2:

Graph the function $f\left(x\right)=\dfrac{2x^2-2}{x^2-4}$ and identify the domain and range.

<table>
<tr><td colspan="2">$f\left(x\right)=\dfrac{2\left(x-1\right)\left(x+1\right)}{\left(x-2\right)\left(x+2\right)}$</td><td>Factor the numerator and the denominator.</td></tr>
<tr><td>$2\left(x-1\right)\left(x+1\right)=0$
$x-1=0 \quad \vert \quad x+1=0$
$x=1 \quad \vert \quad x=-1$
$\left(1,0\right)$ or $\left(-1,0\right)$</td><td>$f\left(0\right)=\dfrac{2\left(0\right)^2-2}{\left(0\right)^2-4}$
$f\left(0\right)=\dfrac{0-2}{0-4}$
$f\left(0\right)=\dfrac{2}{4}=\dfrac{1}{2}$
$\left(0,\dfrac{1}{2}\right)$</td><td>Find the intercepts.</td></tr>
<tr><td>$\left(x-2\right)\left(x+2\right)=0$
$x-2=0 \quad \vert \quad x+2=0$
$x=2 \quad \vert \quad x=-2$</td><td>$y=\dfrac{2}{1}=2$</td><td>Find the asymptotes.

The degree of the numerator (2) is the same as the degree of the denominator (2) so the horizontal asymptote is the ratio of the leading coefficients.</td></tr>
<tr><td colspan="2"></td><td>The sign array determines whether the graph is positive or negative on the interval.</td></tr>
</table>

Intercepts	Asymptotes	Final Sketch
		

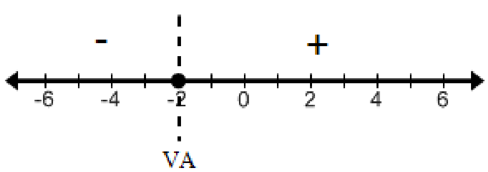
<table>
<tr><td>Domain: $\left(-\infty,-2\right)\cup\left(-2,2\right)\cup\left(2,\infty\right)$

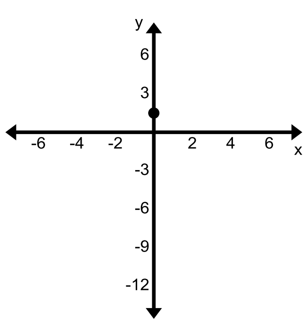
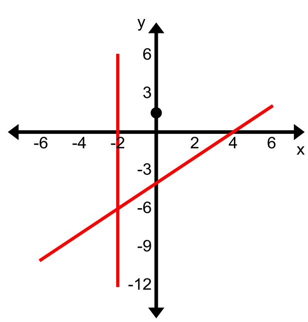
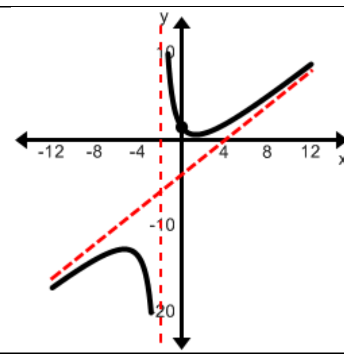
Range: $\left(-\infty,2\right)\cup\left(2,\infty\right)$</td><td>The domain includes all values except for the vertical asymptotes.

The range includes all values except for the horizontal asymptote.</td></tr>
</table>

Example 3:

Graph the function $f(x)=\dfrac{x^2-2x+3}{x+2}$ and identify the domain.

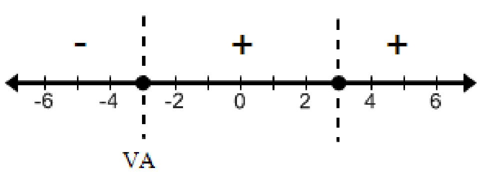
$f(x)=\dfrac{x^2-2x+3}{x+2}$		Factor the numerator and the denominator. The numerator does not factor.
$x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(3)}}{2(1)}$ $x=\dfrac{2\pm\sqrt{4-12}}{2}$ $x=\dfrac{2\pm\sqrt{-8}}{2}$ The solutions are not real.	$f(0)=\dfrac{(0)^2-2(0)+3}{(0)+2}$ $f(0)=\dfrac{3}{2}$ $\left(0,\dfrac{3}{2}\right)$	Find the intercepts. Use the quadratic formula to find the *x*-intercepts.
$x+2=0$ $x=-2$	$\begin{array}{r} x-4 \\ x+2\overline{)\,x^2-2x+3} \\ \underline{-(x^2+2x)} \\ -4x+3 \\ \underline{-(-4x-8)} \\ 11 \end{array}$ $y=x-4$	Find the asymptotes. The degree of the numerator (2) is greater than the degree of the denominator (1) so there may be an oblique asymptote. The quotient is the oblique asymptote, the remainder is disregarded.
		The sign array determines whether the graph is positive or negative on the interval.

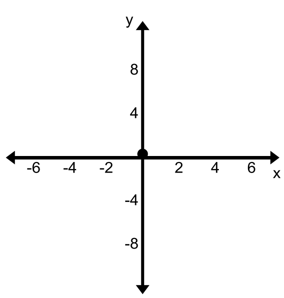
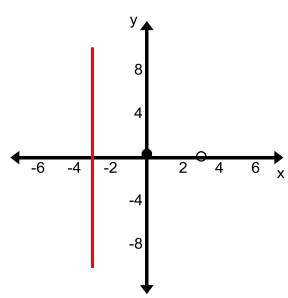
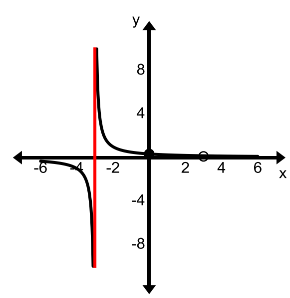
Intercepts	**Asymptotes**	**Final Sketch**
		

Domain: $(-\infty,-2)\cup(-2,\infty)$	The domain includes all values except for the vertical asymptotes.

Example 4:

Graph the function $f(x)=\dfrac{x-3}{x^2-9}$ and identify the domain and range.

$f(x)=\dfrac{x-3}{(x-3)(x+3)}$		Factor the numerator and the denominator.
$x-3=0$ $x=3$ $(3,0)$	$f(0)=\dfrac{0-3}{0^2-9}$ $f(0)=\dfrac{-3}{-9}$ $f(0)=\dfrac{1}{3}$ $\left(0,\dfrac{1}{3}\right)$	Find the intercepts.
$(x-3)(x+3)=0$ $x-3=0$, $x=3$ | $x+3=0$, $x=-3$	$y=0$	Find the asymptotes. Note: Only $x=-3$ is a vertical asymptote because the numerator and the denominator have a common factor. The function is still undefined at $x=3$ because it makes the denominator of the original function zero. The degree of the numerator (1) is less than the degree of the denominator (2) so the horizontal asymptote is $y=0$.
		The sign array determines whether the graph is positive or negative on the interval.

Intercepts	Asymptotes	Final Sketch
		

Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ Range: $(-\infty,0)\cup(0,\infty)$	The domain includes all values except for the values that make the denominator zero. The range includes all values except for the horizontal asymptote.

Practice Exercises A

Graph each rational function and determine the domain and range.

1. $f\left(x\right)=\dfrac{3}{x+7}$

2. $f\left(x\right)=\dfrac{-6}{x+2}$

3. $f\left(x\right)=\dfrac{x-5}{x^2-16}$

4. $f\left(x\right)=\dfrac{x-1}{x^2-9}$

5. $f\left(x\right)=\dfrac{x+7}{x^2-3x-18}$

6. $f\left(x\right)=\dfrac{x-2}{x+1}$

7. $f\left(x\right)=\dfrac{-4x+1}{x-2}$

8. $f\left(x\right)=\dfrac{3x+2}{5x-6}$

9. $f\left(x\right)=\dfrac{x^2+x-30}{3x^2-3}$

10. $f\left(x\right)=\dfrac{2x^2+17x+21}{3x^2+4x-4}$

11. $f\left(x\right)=\dfrac{2x^2-9x+4}{x^2-7x+10}$

12. $f\left(x\right)=\dfrac{x-7}{x^2-4x-21}$

13. $f\left(x\right)=\dfrac{x+4}{x^2-2x-24}$

14. $f\left(x\right)=\dfrac{2x}{x^2+3x}$

15. $f\left(x\right)=\dfrac{x-3}{2x^2-x-15}$

Graph each rational function and determine the domain.

16. $f\left(x\right)=\dfrac{-2}{x^2-6x+8}$

17. $f\left(x\right)=\dfrac{3}{x^2-1}$

18. $f\left(x\right)=\dfrac{3x^2-x}{x+1}$

19. $f\left(x\right)=\dfrac{x^2+x-42}{x-4}$

20. $f\left(x\right)=\dfrac{x^3+8}{x^2-4}$

21. $f\left(x\right)=\dfrac{x^3+1}{x-1}$

Unit 4 Clusters 2 (F.IF.4, and F.IF.5): Key Features of Graphs

Cluster 2: Key features of graphs

4.2 Interpret key features (intercepts, intervals of increasing/decreasing, intervals of positive/negative, relative maximums or minimums, symmetries, and end behavior) of graphs and tables in terms of the quantities.

4.2 Sketch graphs showing key features given a verbal description of the relationship.

4.2 Relate the domain of a function to its graph and the relationship it describes.

VOCABULARY

The ***x*-intercept** is where a graph crosses or touches the x-axis. It is the ordered pair $(a, 0)$. Where a is a real number.

The ***y* -intercept** is where a graph crosses or touches the y-axis. It is the ordered pair $(0, b)$. Where b is a real number.

A **relative maximum** occurs when the *y*-value is greater than all of the *y*-values near it. A function may have more than one relative maximum value. A **relative minimum** occurs when the *y*-value is less than all of the *y*-values near it. A function may have more than one relative minimum value.

An **interval** is a set of numbers between two x-values. An **open interval** is a set of numbers between two x-values that does not include the two end values. **Open intervals** are written in the form (x_1, x_2) or $x_1 < x < x_2$. A **closed interval** is a set of numbers between two x-values that does include the two end values. **Closed intervals** are written in the form $[x_1, x_2]$ or $x_1 \le x \le x_2$.

A function *f* is **increasing** when it is rising (or going up) from left to right and it is **decreasing** when it is falling (or going down) from left to right. A **constant** function is neither increasing nor decreasing; it has the same *y*-value for its entire domain.

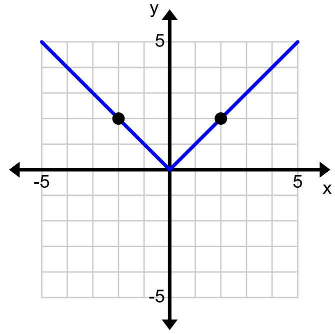
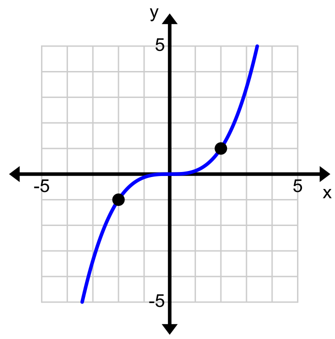
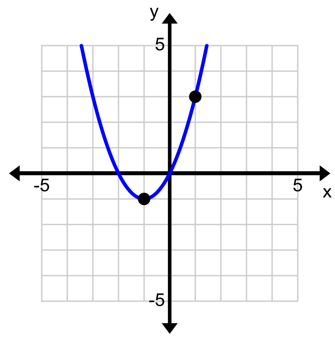
A function is **positive** when $f(x) > 0$ or the *y*-coordinates are always positive. A function is **negative** when $f(x) < 0$ or the *y*-coordinates are always negative.

End behavior describes what is happening to the *y*-values of a graph when *x* goes to the far right $(+\infty)$ or *x* goes the far left $(-\infty)$.

End behavior is written in the following format:

Right End Behavior:
$\lim_{x \to \infty} f(x) = c$

Left End Behavior:
$\lim_{x \to -\infty} f(x) = c$

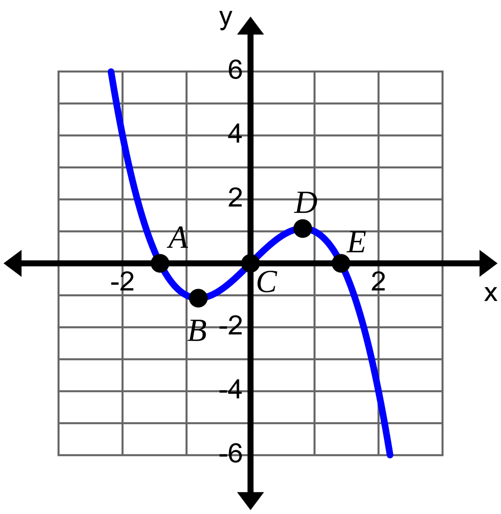
VOCABULARY	GRAPHICALLY	ALGEBRAICALLY
A function is **symmetric with respect to the *y*-axis** if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph. In other words, if you substitute $-x$ in for every x you end up with the original function. When looking at the graph, you could "fold" the graph along the *y*-axis and both sides are the same.		$f(x) = \|x\| + 5$ $f(-x) = \|-x\| + 5$ $f(x) = f(-x) = \|x\| + 5$
A function is **symmetric with respect to the origin** if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. In other words, if you substitute $-x$ in for every x you end up with the opposite of the original function. When looking at the graph, there is a mirror image in Quadrants 1 & 3 or Quadrants 2 & 4.		$f(x) = 8x^3$ $f(-x) = 8(-x)^3$ $f(-x) = -f(x) = -8x^3$
An equation with **no symmetry**. If you substitute $-x$ in for every x you end up with something that is neither the original function nor its opposite. When looking at the graph, you could not "fold" the graph along the *y*-axis and have both sides the same. It also does not reflect a mirror image in opposite quadrants.		$f(x) = x^2 + 2x$ $f(-x) = (-x)^2 + 2(-x)$ $f(-x) = x^2 - 2x \neq f(x) \neq -f(x)$

Example 1:

Analyze the key features of $f(x) = -x^3 + 2x$.

a. Graph the function
b. Identify the intercepts
c. Identify the relative maximums and minimums
d. Identify the intervals where the function is increasing or decreasing
e. Identify the intervals where the function is positive or negative
f. Determine the end behavior
g. Determine the symmetry

a.



b. intercepts: A $(-1.414, 0)$, C $(0, 0)$, E $(1.414, 0)$

c. maximum: D $(0.816, 1.089)$; minimum B $(-0.816, -1.089)$

d. increasing: $(-0.816, 0.816)$; decreasing $(-\infty, -0.816) \cup (0.816, \infty)$

e. positive: $(-\infty, -1.414) \cup (0, 1.414)$; negative: $(-1.414, 0) \cup (1.414, \infty)$

f. right end behavior: $\lim_{x \to \infty} f(x) = -\infty$; left end behavior: $\lim_{x \to -\infty} f(x) = \infty$

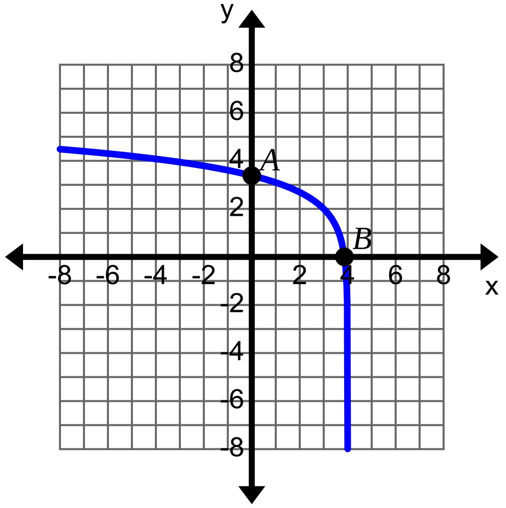
g. odd symmetry

Example 2:

Analyze the key features of $f(x) = \ln(4 - x) + 2$.

a. Graph the function
b. Identify the intercepts
c. Identify the relative maximums and minimums
d. Identify the intervals where the function is increasing or decreasing
e. Identify the intervals where the function is positive or negative
f. Determine the end behavior
g. Determine the symmetry

a.



b. x-intercept: B $(3.865, 0)$;
y-intercept: A $(0, 3.386)$
c. maximum: none
d. increasing: decreasing $(-\infty, 4)$
e. positive: $(-\infty, 3.865)$; negative: $(3.865, 4)$
f. right end behavior: $\lim_{x\to 4^-} f(x) = -\infty$;
left end behavior: $\lim_{x\to -\infty} f(x) = \infty$
g. no symmetry

Practice exercises A

Analyze the key features of $f(x)$.

a. Graph the function
b. Identify the intercepts
c. Identify the relative maximums and minimums
d. Identify the intervals where the function is increasing or decreasing
e. Identify the intervals where the function is positive or negative
f. Determine the end behavior
g. Determine the symmetry

1. $f(x) = \frac{1}{2}|x-3|-5$
2. $f(x) = -2x^2+4$
3. $f(x) = \sqrt[3]{x+2}-3$
4. $f(x) = x^3+x^2-9x-9$
5. $f(x) = -3\sqrt{5-x}+2$
6. $f(x) = e^{x+4}-3$
7. $f(x) = \ln(x-3)-1$
8. $f(x) = 3\sin(-x)$
9. $f(x) = \cos(2x)$
10. $f(x) = \begin{cases} \frac{1}{2}x & \text{if } x<2 \\ (x-4)^2-3 & \text{if } x \ge 2 \end{cases}$
11. $f(x) = -\frac{2}{3}x$
12. $f(x) = \operatorname{int}\ x+2\ +3$

Example 3:
Use the characteristics to sketch a graph of the function described.

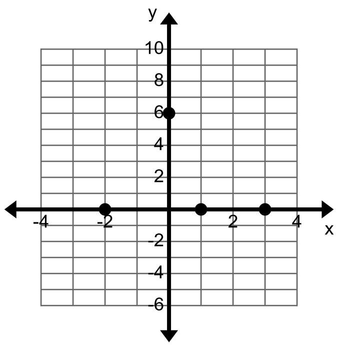
A polynomial function has:

a. x-intercepts: $(-2,0)$, $(1,0)$, and $(3,0)$; y-intercept: $(0,6)$

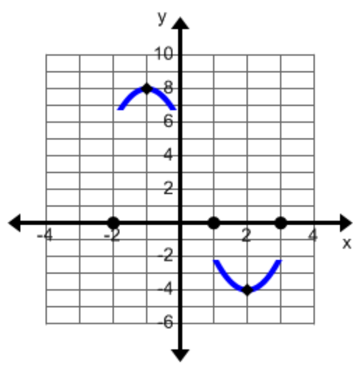
b. relative maximum: $(-1,8)$; relative minimum: $(2,-4)$

c. right end behavior: $\lim\limits_{x\to\infty} f(x)=\infty$; left end behavior: $\lim\limits_{x\to-\infty} f(x)=-\infty$

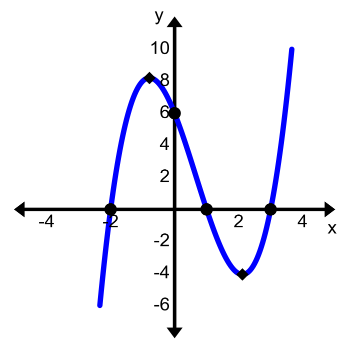
a. Plot the intercepts.



b. Plot the maximum and minimums



c. Draw the end behavior in and connect the points.



Example 4:
Use the characteristics to sketch a graph of the function described.

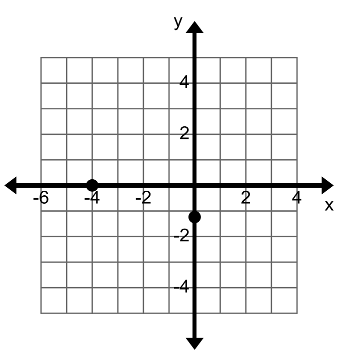
A square root function has:

a. x-intercept: $(-4,0)$; y-intercept: $(0,-1.236)$

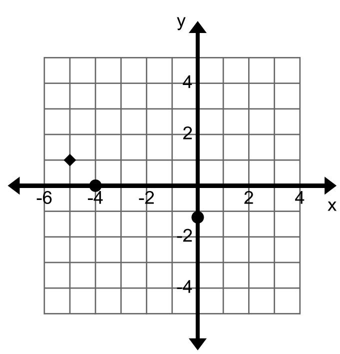
b. relative maximum: $(-5,1)$

c. right end behavior: $\lim\limits_{x\to\infty} f(x)=-\infty$; left end behavior: $\lim\limits_{x\to-5^{+}} f(x)=1$

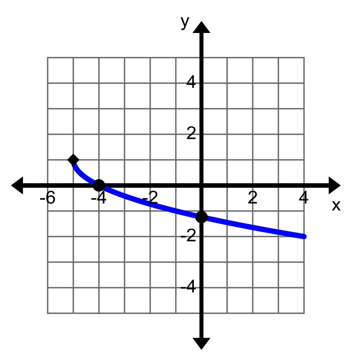
a. Plot the intercepts.



b. Plot the maximum.



c. Draw the end behavior in and connect the points.



Practice Exercises B

Use the characteristics to sketch a graph of the function described.

1. $f(x)$ is an even function that decreases at a constant rate from $(-\infty, 0)$, has a minimum at $(0,-3)$ and a point at $(2,0)$.

2. $g(x)$ is a periodic, odd function that has x-intercepts at $(0,0)$ and $(\pm 4k, 0)$, a maximum at $(2,3)$ and a minimum at $(6,-3)$.

3. $h(x)$ is a function with a maximum value at $(1,9)$, intercepts at $(-2,0)$ and $(4,0)$, and end behavior $\lim_{x\to\infty} h(x) = -\infty$ and $\lim_{x\to-\infty} h(x) = -\infty$.

4. $j(x)$ is a function with an intercepts at $(-1,0)$ and $(0,3)$, end behavior $\lim_{x\to-2^+} j(x) = -\infty$ and $\lim_{x\to\infty} h(x) = \infty$, and passes through the point $(3,7)$.

5. $f(x)$ is a function that increases $(-\infty, \infty)$ with intercepts at $(4,0)$ and $(0,-6)$, and end behavior $\lim_{x\to\infty} h(x) = \infty$ and $\lim_{x\to-\infty} h(x) = -9$.

6. $g(x)$ is an even function with a minimum at $(-2,-5)$, a maximum at $(0,0)$, intercepts at $(-3,0)$ and $(3,0)$, and end behavior $\lim_{x\to\infty} h(x) = \infty$.

7. $h(x)$ is a function with intercepts at $(-3,0)$, $(1,0)$, and $(0,-3)$, minimum at $(-1,-4)$ and maximums at $(-3,0)$ and $(1,0)$, and end behavior $\lim_{x\to\infty} h(x) = -\infty$ and $\lim_{x\to-\infty} h(x) = -\infty$.

8. $j(x)$ is a function with intercepts at $(-5,0)$, $(-2,0)$, $(0,0)$, $(2,0)$, and $(4,0)$, maximums at $(-4,38)$ and $(1,5)$, minimums at $(-1,-6)$ and $(3,-12)$, and end behavior $\lim_{x\to\infty} h(x) = \infty$ and $\lim_{x\to-\infty} h(x) = -\infty$.

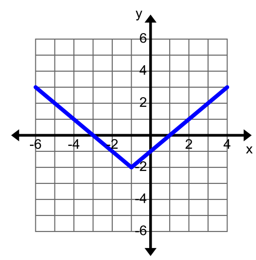
Domain

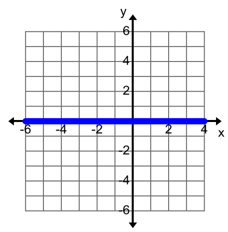
VOCABULARY
The **domain** is the set of all first coordinates when given a table or a set of ordered pairs. It is the set of all *x*-coordinates of the points on the graph and is the set of all numbers for which a function is defined. The domain it is written from the least value to the greatest value.

Example 5:
Find the domain of $f\left(x\right)=2\sqrt{x+2}-3$.

1. Find any values for which the function is undefined.	The square root function has real number solutions if the expression under the radicand is positive or zero. This means that $x+2\geq 0$ therefore $x\geq -2$.
2. Write the domain in interval notation.	The domain is $[-2,\infty)$.

Example 6:
Find the domain of the function graphed to the right.



1. List all the *x*-values of the function graphed.	If you were to flatten the function against the *x*-axis you would see something like this:  The function is defined for all the *x*-values along the *x*-axis.
2. Write the domain in interval notation.	The domain is $\left(-\infty,\infty\right)$.

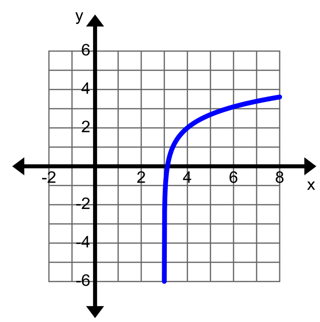
Example 7:

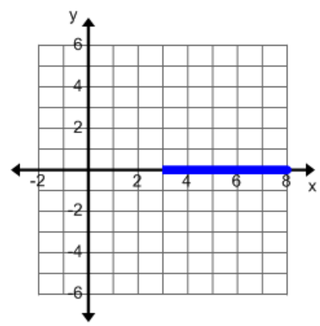
Find the domain of $f\left(x\right)=\dfrac{x+4}{x^2-3x-4}$.

1. Find any values for which the function is undefined. $x^2-3x-4=0$ $\left(x-4\right)\left(x+1\right)=0$ $x-4=0$ or $x+1=0$ $x=4$ or $x=-1$	Division by zero is undefined, so exclude from the domain any values that would make the denominator zero.
2. Write the domain in interval notation.	The domain is $\left(-\infty,-1\right)\cup\left(-1,4\right)\cup\left(4,\infty\right)$.

Example 8:

Find the domain of $f\left(x\right)=\ln\left(x-3\right)+2$ graphed to the right.

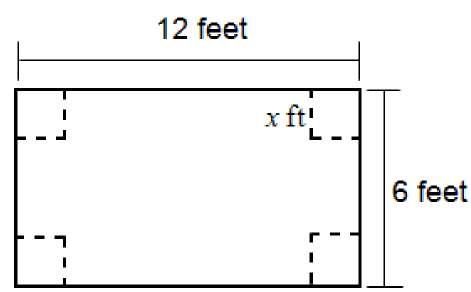


1. List all the *x*-values of the function graphed.	If you were to flatten the function against the *x*-axis you would see something like this:  Notice that the vertical asymptote has been shifted to $x=3$. The function is defined for all the *x*-values along the *x*-axis from 3 to infinity.
2. Write the domain in interval notation.	The domain is $\left(3,\infty\right)$.

Example 9:
The path of a ball thrown straight up can be modeled by the equation $h(t) = -16t^2 + 48t + 12$ where *t* is the time in seconds that the ball is in the air and *h* is the height of the ball. What is the real world domain for the situation?

1. Find all the values that would make sense for the situation.	The domain represents the amount of time that the ball is in the air. At $t = 0$ the ball is thrown and enters the air shortly afterwards so the domain must be greater than zero. The ball will hit the ground at 3.232 seconds. Once it is on the ground it is no longer in the air so the domain must be less than 3.232 seconds. The ball is in the air for $0 < t < 3.232$ seconds.
2. Write the domain in interval notation.	The domain is $(0, 3.232)$.

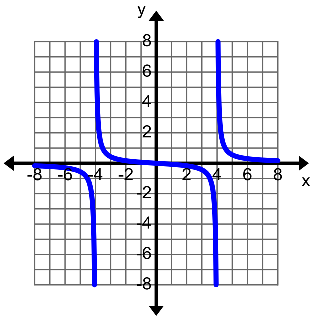
Example 10:
A square of side *x* feet is cut out of each corner of a 12 feet by 6 feet piece of material to form an open-topped box. Determine the domain of the volume function in terms of *x*.

1. Write a function for the volume in terms of *x*. The height is $h = x$. The length is $l = 12 - 2x$. The width is $w = 6 - 2x$. The volume is $V = lwh$. $V(x) = x(12-2x)(6-2x)$	
2. Find all the values that would make sense for the situation. $6 - 2x = 0$ $6 = 2x$ $3 = x$	The length of a side cannot be zero. Therefore, take the shorter side and set it equal to zero. The domain must be values that are less than this value but greater than zero.
3. Write the domain in interval notation.	The domain is $(0, 3)$.

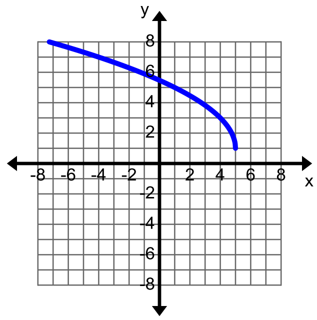
Practice Exercises C

Find the domain.

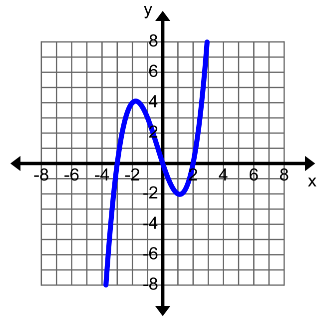
1.



2.



3.



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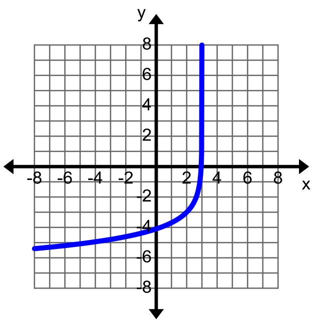
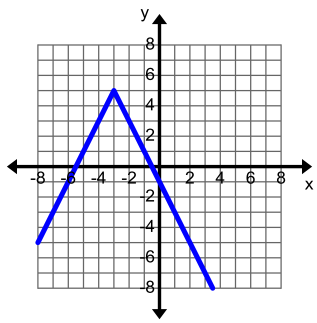
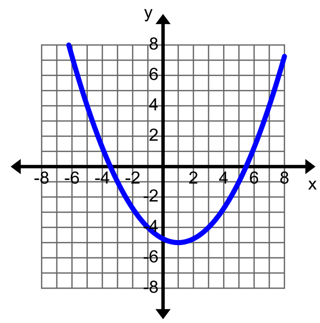


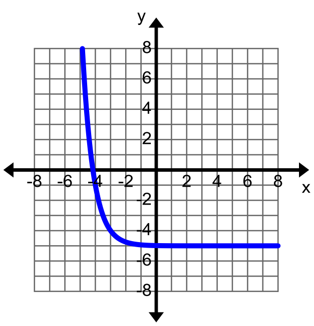
5.



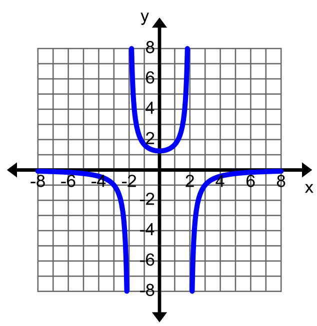
6.



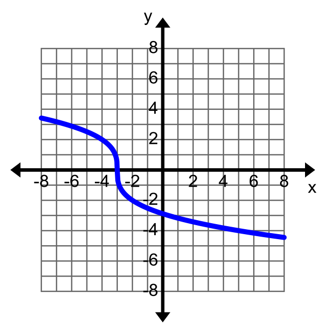
7.



8.



9.



10. $f(x) = 1 - 2\sqrt{3-x}$

11. $f(x) = \dfrac{2x+1}{x^2+3x-10}$

12. $f(x) = -2\log(x+2)+1$

13. $f(x) = 4\sqrt[3]{-\dfrac{1}{2}x}$

14. $f(x) = (x-2)^3 - 4$

15. $f(x) = \sin(\pi x)+3$

16. $f(x) = \dfrac{8x-3}{4x+5}$

17. $f(x) = 2|x+4|-5$

18. $f(x) = \sqrt{x-4}$

19. A square of side x inches is cut out of each corner of a 30 inches by 24 inches piece of material to form an open-topped box. Determine the domain of the volume function in terms of x.

20. An object is dropped from the ledge of an open window that is 25 feet above the ground. What is the domain for the situation? (Use $h(t) = -16t^2 + 25$.)

21. A garden that is 20 feet by 15 feet is being reduced on each side by x feet and having a brick border placed around the reduced garden. Determine the domain of the area function that describes the reduced garden in terms of x.

22. A parking garage charges \$1.50 per hour with a maximum charge of \$12 per day. Determine the domain.

Unit 4 Cluster 3 (F.IF.9): Comparing Functions

Cluster 3: Analyze Functions using Different Representations

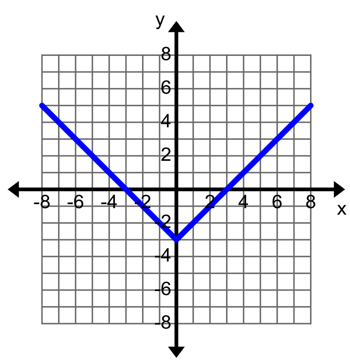
4.3 Compare properties (key features) of functions each represented differently (table, graph, equation or description)

Example 1:

Compare the properties of function A to those of function B.

a. *x* and *y*-intercepts
b. intervals of increasing or decreasing
c. intervals of positive or negative
d. maximums or minimums
e. domain and range
f. end behavior

Function A



Function B

$$y = x - 3$$

Function A	Function B
a. *x*-intercepts $(-3,0)$ and $(3,0)$; *y*-intercept $(0,-3)$	a. *x*-intercept $(3,0)$; *y*-intercept $(0,-3)$
b. increasing $(0,\infty)$; decreasing $(-\infty,0)$	b. increasing $(-\infty,\infty)$
c. positive $(-\infty,-3)\cup(3,\infty)$; negative $(-3,3)$	c. positive $(3,\infty)$; negative $(-\infty,3)$
d. minimum $(0,-3)$	d. none
e. domain $(-\infty,\infty)$; range $[-3,\infty)$	e. domain $(-\infty,\infty)$; range $(-\infty,\infty)$
f. $\lim_{x\to\infty} f(x)=\infty$; $\lim_{x\to-\infty} f(x)=\infty$	f. $\lim_{x\to\infty} f(x)=\infty$; $\lim_{x\to-\infty} f(x)=-\infty$

On the interval $(0,\infty)$ both functions have the same characteristics. However, on the interval $(-\infty,0)$ there are quite a few differences: function A has an additional *x*-intercept, function A has a minimum, function A is decreasing on the interval, function A is positive on part of the interval, and the left end behavior is different.

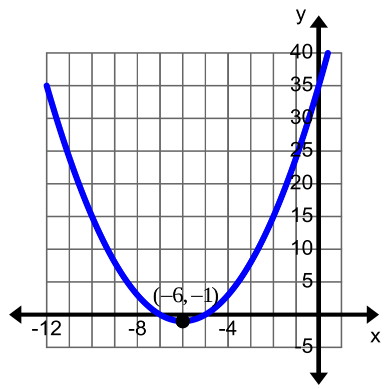
Example 2:
Compare the properties of function A to those of function B.

a. x and y-intercepts
b. intervals of increasing or decreasing
c. intervals of positive or negative
d. maximums or minimums
e. domain and range

Function A

x	$f(x)$
0	35
3	8
4	3
5	0
6	-1
7	0
8	3
9	8

Function B



Function A	Function B
a. x-intercepts $(5,0)$ and $(7,0)$; y-intercept $(0,35)$	a. x-intercepts $(-7,0)$ and $(-5,0)$; y-intercept $(0,35)$
b. increasing $(6,\infty)$; decreasing $(-\infty,6)$	b. increasing $(-6,\infty)$; decreasing $(-\infty,-6)$
c. positive $(-\infty,5)\cup(7,\infty)$; negative $(5,7)$	c. positive $(-\infty,-7)\cup(-5,\infty)$; negative $(-7,-5)$
d. minimum $(6,-1)$	d. minimum $(-6,-1)$
e. domain $(-\infty,\infty)$; range $[-1,\infty)$	e. domain $(-\infty,\infty)$; range $[-1,\infty)$

Both functions have the same y-intercept, minimum value, domain, range, and end behavior. The points forthe minimum and the x-intercepts are reflections of one another over the y-axis.

Example 3:
Compare the properties of function A to those of function B.

a. intercepts
b. intervals of increasing or decreasing
c. intervals of positive or negative
d. maximums or minimums
e. symmetry

<table>
<tr><th>Function A</th><th>Function B</th></tr>
<tr><td>$f(x) = x^3 - 4x$</td><td>
<table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>-3</td><td>-192</td></tr>
<tr><td>-2</td><td>-48</td></tr>
<tr><td>-1</td><td>0</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>2</td><td>48</td></tr>
<tr><td>3</td><td>192</td></tr>
</table>
Relative minimum $\left(\sqrt{\frac{1}{3}}, -3.079\right)$
Relative Maximum $\left(-\sqrt{\frac{1}{3}}, 3.079\right)$
</td></tr>
</table>

Function A	Function B
a. intercepts $(-2,0)$, $(0,0)$ and $(2,0)$	a. intercepts $(-1,0)$, $(0,0)$ and $(1,0)$
b. increasing $\left(-\infty, -2\sqrt{\frac{1}{3}}\right) \cup \left(2\sqrt{\frac{1}{3}}, \infty\right)$; decreasing $\left(-2\sqrt{\frac{1}{3}}, 2\sqrt{\frac{1}{3}}\right)$	b. increasing $\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right)$; decreasing $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$
c. positive $(-2,0) \cup (2,\infty)$; negative $(-\infty,-2) \cup (0,2)$	c. positive $(-1,0) \cup (1,\infty)$; negative $(-\infty,-1) \cup (0,1)$
d. minimum $\left(2\sqrt{\frac{1}{3}}, -3.079\right)$; maximum $\left(-2\sqrt{\frac{1}{3}}, 3.079\right)$	d. minimum $\left(\sqrt{\frac{1}{3}}, -3.079\right)$; maximum $\left(-\sqrt{\frac{1}{3}}, 3.079\right)$
e. odd symmetry	e. odd symmetry

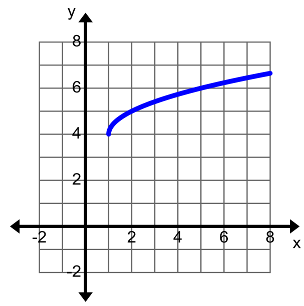
The functions have the same relative minimum and maximum value, but it occurs in different places. They both have the same *y*-intercept and they both have odd symmetry. It would seem that function A has twice the width as function B, but they seem to behave the same way between intercepts and maximums and minimums.

Practice Exercises A

1. Compare the properties of function A to those of function B.

   a. intervals of increasing or decreasing
   b. intervals of positive or negative
   c. maximums or minimums
   d. domain and range

Function A:



Function B:

$y=-\sqrt{x-1}+4$

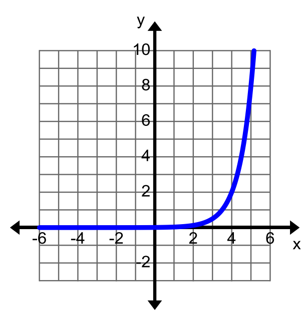
2. Compare the properties of function A to those of function B.

   a. y-intercept
   b. average rate of change on the interval $[4,5]$
   c. intervals of positive or negative
   d. domain and range

Function A:

x	$f(x)$
0	$\frac{1}{64}$
1	$\frac{1}{16}$
2	$\frac{1}{4}$
3	1
4	4
5	16

Function B:



3. Compare the properties of function A to those of function B.

   a. x-intercepts and y-intercepts
   b. maximums or minimums
   c. range
   d. symmetry

Function A:

x	$f(x)$
-6	0
-5	2
-3	4
-1	2
0	0
1	-2
3	-4
5	-2
6	0

Function B:

$$f(x) = 4\sin\left(\frac{\pi}{6}x\right)$$

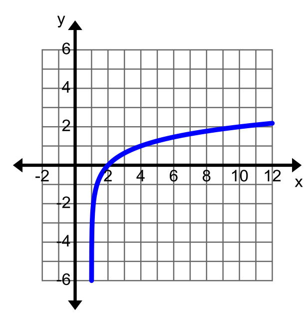
4. Compare the properties of function A to those of function B.

   a. x-intercept
   b. intervals of increasing or decreasing
   c. intervals of positive or negative
   d. domain

Function A:

x	$f(x)$
1	Undefined
$1.\overline{1}$	0
2	2
4	3
6	3.465
8	3.771
10	4

Function B:



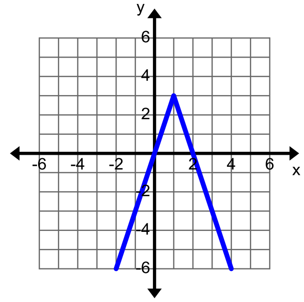
5. Compare the properties of function A to those of function B.

   a. x-intercepts and y-intercepts
   b. intervals of increasing and decreasing
   c. maximums or minimums
   d. domain and range

Function A:

$$f(x) = -3|x-1| + 6$$

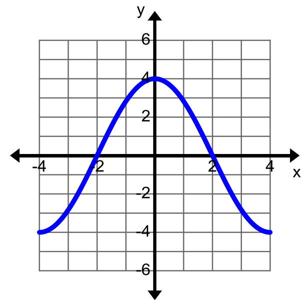
Function B:



6. Compare the properties of function A to those of function B.

   a. x-intercepts and y-intercepts
   b. maximums or minimums
   c. range
   d. symmetry

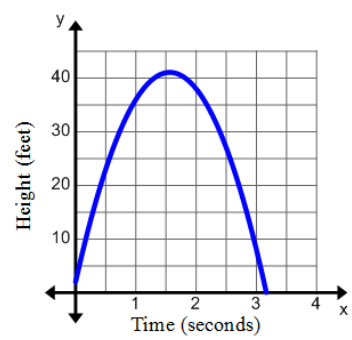
Function A:



Function B:

x	$f(x)$
-2	-4
-1	0
0	4
1	0
2	-4

7. A water powered rocket can be launched from a 0.16 foot platform straight up into the air with an initial velocity of 47 feet per second. A similar water powered rocket's height has been graphed at the right. Which rocket is in the air the longest? Which rocket has the greatest maximum height?



8. Compare the properties of function A to those of function B.
   a. x and y-intercepts
   b. intervals of increasing or decreasing
   c. intervals of positive or negative
   d. maximums or minimums

Function A:
$f(x)=(x-2)^3-1$

Function B:

x	$f(x)$
-4	-9
-3	-2
-2	-1
-1	0
0	7
1	26
2	63

Unit 4 Cluster 2(F.IF.6) Average Rate of Change

Cluster 2: Interpret Functions that Arise in Applications in Terms of a Context

4.2 Calculate, interpret, and estimate from a graph the average rate of change over an interval. Include rational, square root, cube root, polynomial, logarithmic, and trigonometric functions in addition to quadratic and exponential.

VOCABULARY

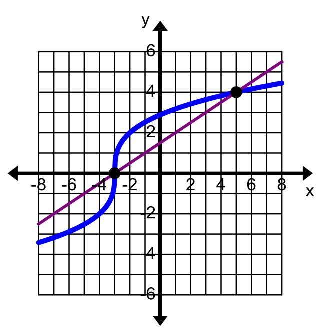
The **average rate of change** of a function over an interval is the ratio of the difference (change) in *y* over the difference (change) in *x*.

$$\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The line connecting the two points is called the **secant line**.

Example 1:

Find the average rate of change for $f(x) = 2\sqrt[3]{x+3}$ on the interval $[-3, 5]$.

<table>
<tr><td>$f(-3) = 2\sqrt[3]{-3+3}$
$f(-3) = 2\sqrt[3]{0}$
$f(-3) = 0$</td><td>$f(5) = 2\sqrt[3]{5+3}$
$f(5) = 2\sqrt[3]{8}$
$f(5) = 4$</td><td>First, find the value of the function at each end point of the interval.</td></tr>
<tr><td colspan="2">
$m = \frac{4-0}{5-(-3)} = \frac{4}{8} = \frac{1}{2}$</td><td>Next, find the slope between the two points $(-3, 0)$ and $(5, 4)$.</td></tr>
<tr><td colspan="3">The average rate of change of $f(x) = 2\sqrt[3]{x+3}$ on the interval $[-3, 5]$ is $\frac{1}{2}$.</td></tr>
</table>

Example 2:
The table shows the total US farm exports in billions for several years. Find the average amount per year from 1996 to 2000.

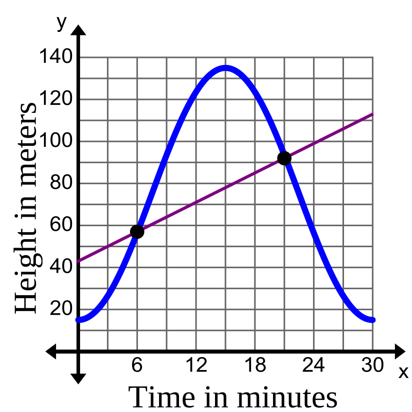
Years	Amount (billions)
1980	41.2
1985	29.0
1990	39.5
1992	43.2
1993	42.9
1994	46.3
1995	56.3
1996	60.4
1997	57.2
1998	51.8
1999	48.5
2000	51.6

The year 1996 is 16 years since 1980 and 2000 is 20 years since 1980, therefore the interval is [16, 20]. Find the slope between the two points (16, 60.4) and (20, 51.6).

$$m = \frac{51.6 - 60.4}{20 - 16} = \frac{-8.8}{4} = -2.2$$

The average rate of change from 1996 to 2000 is -2.2 billions of exports each year. This means that the number of exports decreases about 2.2 billion each year between 1996 and 2000.

Example 3:
Jane is visiting London and took a ride on the London Eye. Her distance in meters from the ground at any given time is shown in the graph at the right. Find her average rate of change from 6 to 21 minutes.



At 6 minutes her height appears to be close to 60 meters and at 21 minutes her height appears to be 90 meters. Find the slope between the two points (6, 60) and (21, 90).

$$m = \frac{90-60}{21-6} = \frac{30}{15} = 2$$

Jane's average rate of change is 2 meters per minute. This means that she is traveling at an average rate of 2 meters per minute from 6 to 21 minutes.

Practice Exercises A

Find the average rate of change for each function on the specified interval.

1. $f(x) = 3x^2 - x + 5$ on [-1, 3]
2. $f(x) = 4x^2 + 12x + 9$ on [-3, 0]
3. $f(x) = -x^2 + 4$ on $[3, a]$
4. $f(x) = \dfrac{x-7}{x^2+14x+40}$ on $[-9,-5]$
5. $f(x) = \dfrac{x^2+11x+30}{x+6}$ on $[-4,0]$
6. $f(x) = \dfrac{x^2+x-72}{x^2+5x}$ on $[-4,-1]$
7. $f(x) = \sqrt{x+8} - 6$ on $[-4,1]$
8. $f(x) = \sqrt{x-7} + 4$ on $[7,11]$
9. $f(x) = 2\sqrt{x+3} - 10$ on $[1,6]$
10. $f(x) = \sqrt[3]{x} - 6$ on $[-1,1]$
11. $f(x) = \sqrt[3]{x+2} - 1$ on $[-3,6]$
12. $f(x) = -\sqrt[3]{x+6} + 2$ on $[-5,2]$
13. $f(x) = x^3 + x - 2$ on $[-3,2]$
14. $f(x) = x^4 - 8x^3 + 16x^2$ on $[-1,3]$
15. $f(x) = x^3 - 9x$ on $[-2,2]$
16. $f(x) = \log(x-3) - 4$ on $[4,13]$
17. $f(x) = 2\ln(x+1) - 3$ on $[0,3]$
18. $f(x) = \ln(-x+2) - 5$ on $[-7,1]$
19. $f(x) = 4\sin x + 7$ on $\left[-\pi, \dfrac{\pi}{2}\right]$
20. $f(x) = -2\cos x - 3$ on $[0,\pi]$
21. $f(x) = 3\sin x - 3$ on $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$
22. $f(x) = 4^{x-1} - 5$ on $[-1,3]$
23. $f(x) = -2 \cdot 3^x + 4$ on $[1,3]$
24. $f(x) = \dfrac{1}{2}\left(\dfrac{1}{4}\right)^x + 2$ on $[-5,-1]$

Find the average rate of change on the specified interval and interpret its meaning.

25. The average temperature per month is shown in the table below. Find the average rate of change from March to October.

Month	Temperature °F
January	34
February	30
March	39
April	44
May	58
June	67
July	78
August	80
September	72
October	63
November	51
December	40

26. The table shows average annual consumption of cheese per person in the U.S. for selected years. (Source: U.S. Department of Agriculture). What is the average consumption between 1940 and 1995.

Year	Pounds Consumed
1910	4
1940	5
1970	8
1975	10
1995	25
2001	30

27. The table below shows the percentage of the U.S. labor force in unions for selected years between 1955 and 2005. Find the average rate of change from 1975 to 1995.

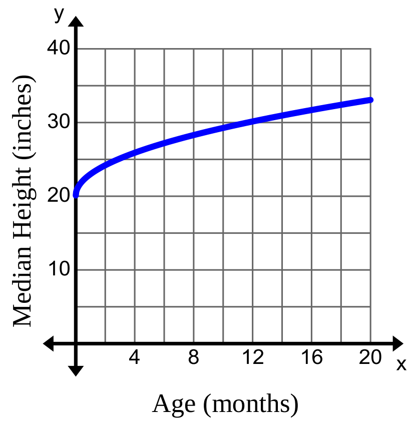
Year	Percent
1955	33.2
1960	31.4
1965	28.4
1970	27.3
1975	25.5
1980	21.9
1985	18.0
1990	16.1
1995	14.9
2000	13.5
2005	12.5

28. The table below shows the amount of carbon dioxide in the Earth's atmosphere for selected years. (Source: the Weather Channel.) Find the average rate of change from 1968 to 2003.

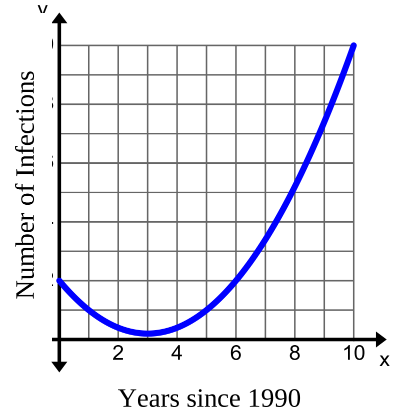
Year	CO_2 in Atmosphere (ppm)
1968	324.14
1983	343.91
1998	367.68
2003	376.68
2008	385.60

Find the average rate of change and interpret its meaning.

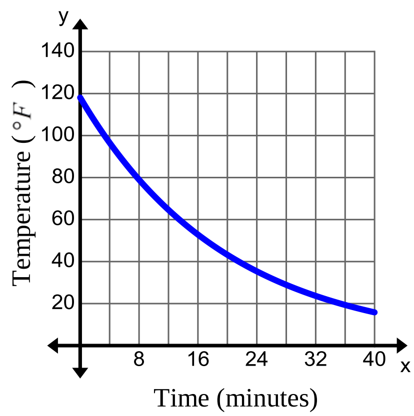
29. The graph below represents the height in inches of boys age x months. Find the average rate of change from 6 months to 16 months.



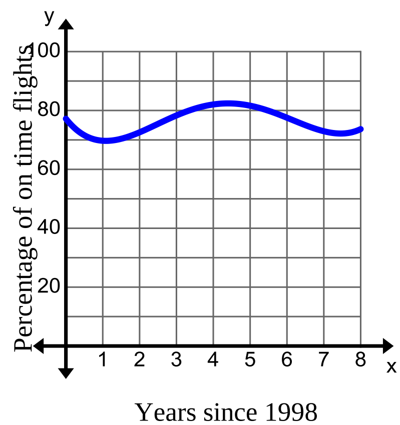
30. The graph below displays the number of infections per month for every 1,000 computers since 1990. Find the average rate of change from 1991 to 1998.



31. A cup of hot liquid is left out to cool. The graph below displays its temperature over time. Find the average rate of change from 4 to 12 minutes.



32. The graph below shows the percentage of on time flights per year since 1998. Find the average rate of change from 1999 to 2003.



Unit 2 Cluster 9 (A.REI.11): Solving Systems of Equations Graphically

Cluster 9: Represent and solve equations and inequalities graphically.

2.9 Explain why the *x*-coordinate of the points where the graphs intersect are solutions of the equation.

2.9 Find the solutions approximately using technology (linear, polynomial, rational, absolute value, exponential, and logarithmic functions).

Example 1:

Using the tables below, find when $f(x) \approx g(x)$ if $f(x) = x^2 - 3$ and $g(x) = -\frac{1}{2}x + 1$.

x	$f(x)$
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6

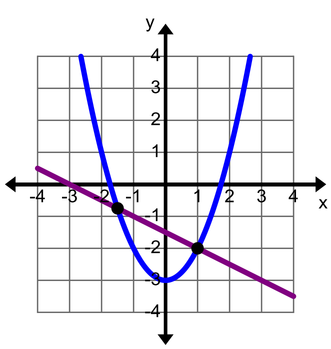
x	$g(x)$
-3	0
-2	-0.5
-1	-1
0	-1.5
1	-2
2	-2.5
3	-3

x	$f(x)$
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6

x	g(x)
-3	0
-2	-0.5
-1	-1
0	-1.5
1	-2
2	-2.5
3	-3

It is obvious that the functions have the same *y*-value when $x = 1$, which means that the two functions intersect when $x = 1$.

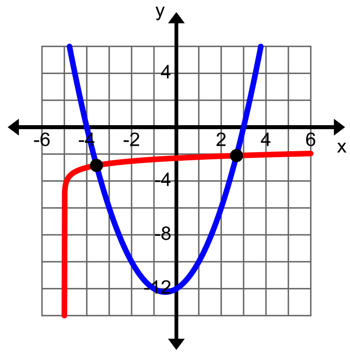
However, it is more difficult to see the second intersection. Notice that on the interval $-2 \le x \le -1$ the value of $f(x)$ is between 1 and -2. Similarly, on the interval $-2 \le x \le -1$ the value of $g(x)$ is between -0.5 and -1. Since both functions are continuous, they take on every value between 1 and -2 for $f(x)$ and between -0.5 and -1 for $g(x)$ which means that they will have the same *y*-value at some *x*-value between -2 and -1.



To find the second intersection more accurately, use technology to graph the functions and find the intersection. The intersection is $(-0.5, -0.75)$ so the functions are the same value when $x = -0.5$.

Example 2:

Use technology to find when $f(x)=g(x)$ if $f(x)=x^2+x-12$ and $g(x)=\log(x+5)-3$.



Graph both functions and find the intersection(s).

Note: There are two logarithmic buttons on your calculator. The common logarithm (log) which is base 10 and the natural logarithm (ln) which is base e.

$f(x)$ and $g(x)$ are the same value when $x=-3.567$ and $x=2.684$.

Practice Exercises A

Use the tables to find when $f(x)\approx g(x)$.

1. $f(x)=x^2+x+4$, $g(x)=2x+6$

x	$f(x)$
-3	10
-2	6
-1	4
0	4
1	6
2	10
3	16

x	g(x)
-3	0
-2	2
-1	4
0	6
1	8
2	10
3	12

2. $f(x)=3x^2+2x-18$, $g(x)=-\frac{1}{2}x-1$

x	$f(x)$
-4	22
-3	3
-2	-10
-1	-17
0	-18
1	-13
2	-2

x	g(x)
-4	1
-3	0.5
-2	0
-1	-0.5
0	-1
1	-1.5
2	-2

3. $f(x)=x^2-14$, $g(x)=|x+2|-10$

x	$f(x)$
-3	-5
-2	-10
-1	-13
0	-14
1	-13
2	-10
3	-5

x	g(x)
-3	-9
-2	-10
-1	-9
0	-8
1	-7
2	-6
3	-5

4. $f(x)=5x+2$, $g(x)=|x-4|-2$

x	$f(x)$
-3	-13
-2	-8
-1	-3
0	2
1	7
2	12
3	17

x	g(x)
-3	5
-2	4
-1	3
0	2
1	1
2	0
3	-1

Practice Exercises B

Use technology to find when $f(x)=g(x)$.

1. $f(x)=-2x+4$
   $g(x)=x^2+3$

2. $f(x)=\frac{1}{3}x-2$
   $g(x)=x^3-3x^2-4x$

3. $f(x)=-\frac{5}{2}x+3$
   $g(x)=\frac{x-1}{2x+1}$

4. $f(x)=-\frac{3}{4}x+7$
   $g(x)=-|x-5|+4$

5. $f(x)=4x-3$
   $g(x)=3^{x-2}-5$

6. $f(x)=-\frac{1}{10}x+3$
   $g(x)=\log(x+2)$

7. $f(x)=(x-5)^2-3$
   $g(x)=\frac{1}{5}x^3-\frac{12}{5}x^2+7x$

8. $f(x)=x^2+12x+31$
   $g(x)=\frac{x-4}{x^2-4x+4}$

9. $f(x)=(x-2)^2-4$
   $g(x)=|x-1|+3$

10. $f(x)=2x^2-7$
    $g(x)=5^{x-2}-3$

11. $f(x)=x^2-6x-7$
    $g(x)=\log(x+1)+3$

12. $f(x)=|2x-1|-3$
    $g(x)=\frac{1}{2}x^3-\frac{3}{2}x^2-\frac{13}{2}x+\frac{15}{2}$

13. $f(x)=\frac{2}{3}|x+1|-8$
    $g(x)=\frac{2x^2}{x^2-4x+45}$

14. $f(x)=|x+4|+3$
    $g(x)=3^{x-2}+1$

15. $f(x)=|x+6|$
    $g(x)=\log(5-x)+3$

Solving Equations and Inequalities

Unit 4 Cluster 1 (A.CED.1, A.SSE.2, and A.CED.4): Writing and Solving Equations and Inequalities in One Variable

Cluster 1: Create Equations that describe numbers or relationships

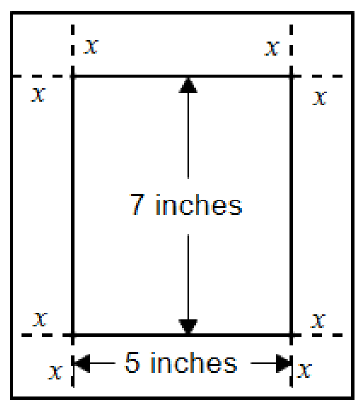
4.1 Create equations in one variable and use them to solve problems (include simple rational, square root, and polynomial)
4.1 Create inequalities in variable and use them to solve problems (include simple rational, square root, and polynomial)
4.1 Rearrange formulas to highlight a quantity of interest, using the same reasoning as solving equations.
2.2 Use the structure of an expression to rewrite it.

When solving contextual type problems it is important to:

- Identify what you know.
- Determine what you are trying to find.
- Draw a picture to help you visualize the situation when possible. Remember to label all parts of your drawing.
- Use familiar formulas to help you write equations.
- Check your answer for reasonableness and accuracy.
- Make sure you answered the entire question.
- Use appropriate units.

Example 1:

You want to create a custom border for a picture of you and your closest friends. The picture measures 5 inches by 7 inches. What should the width of the border be if the final area, including the border, is twice the area of the picture?



$A = lw$ $2(7\cdot 5) = (7+2x)(5+2x)$	The area of a rectangle is the product of the length and width. The length is $l = 7+2x$. The width is $w = 5+2x$. You want the final area to be twice the area of the picture.
$70 = 4x^2 + 24x + 35$ $0 = 4x^2 + 24x - 35$	Simplify the expression and make sure the equation is equal to zero.
$x = 1.2130749$ The border should be about 1.2 inches wide.	Use the quadratic formula or technology to find the zero.

Example 2:
The height of a plastic rectangular prism storage container is 4 inches shorter than the width. The length is 7 inches longer than the width. The volume of the storage container is 5304 square inches. What are the dimensions of the container?

$h=w-4$ $l=w+7$	Let w represent the width of the box. Write equations for the height and the length in terms of the width.
$V=lwh$ $5304=(w+7)(w)(w-4)$	The volume of a rectangular prism is $V=lwh$. Substitute in the values for the length , height, and volume of the box.
$5304=w^3+3w^2-28w$	Expand the right side of the equation.
$0=w^3+3w^2-28w-5304$ $w=17$	Make sure the equation is equal to zero then use technology to find the zeros.
The width is 17 inches, the length is 24 inches, and the height is 13 inches.	$h=17-4=13$ $l=17+7=24$

Work Problems
The equation $\frac{t}{a}+\frac{t}{b}=1$, where a is the amount of time for A to complete the work alone, b is the amount of time for B to complete the work alone, and t is the amount of time needed for A and B to complete the work together, can be used to find the amount of time required for work to be done.

Example 3:
Britton can refinish the floor in 9 hours. Britton and Jason can refinish the floor together in 4 hours. How long would it take Jason to finish the floors himself?

$\frac{t}{a}+\frac{t}{b}=1$	Use the work formula.
$\frac{4}{9}+\frac{4}{b}=1$	Substitute the values you know. Let $a=9$ and $t=4$.
$\frac{4}{b}=\frac{5}{9}$ $36=5b$ $\frac{36}{5}=b$ $7.2=b$	Solve for b.
It would take Jason 7 hours and 12 minutes to refinish the floors by himself.	0.2 of an hour (60 minutes) is 12 minutes. $0.2\times 60=12$

Example 4:
A tugboat goes 12 mph in still water. It travels 45 miles upstream and 45 miles back in 8 hours. What is the speed of the current?

$\frac{d}{r_1}+\frac{d}{r_2}=t$ $\frac{45}{12-c}+\frac{45}{12+c}=8$	Recall that $d=rt$. We know the total time so solving this equation for time yields $\frac{d}{r}=t$. The total time for the trip was 8 hours. Traveling upstream, the boat moves against the current so the rate is $r=12-c$. Coming back the boat moves with the current so the rate is $r=12+c$.
$\left(\frac{45}{12-c}+\frac{45}{12+c}=8\right)\left(144-c^2\right)$ $45(12+c)+45(12-c)=1152-8c^2$ $540+45c+540-45c=1152-8c^2$ $1080=1152-8c^2$	Multiply by the LCM $(12-c)(12+c)=144-c^2$. Then simplify.
$-72=-8c^2$ $9=c^2$ $\pm 3=c$	Solve for *c*.
The speed of the current will not be negative so the current is 3 mph.	

Practice Exercises A

1. An open box is made from a rectangular piece of cardboard measuring 12 inches by 16 inches by cutting identical squares from the corners and turning up the sides. What are the lengths of the sides of the removed squares if the area of the bottom of the open box is 60 square inches?

2. A triangular table top has a base that is twice as long as its height. If the area of the table surface is 324 square inches, what is the value of the height and the base?

3. A family had three children and were expecting a fourth. The oldest was 3 years older than the youngest. The youngest was one year younger than the middle child. How old was each of the children on the day their new sibling was born if the product of their ages was 987 more than three times the sum of their ages.

4. The width of a box is two inches less than twice the height. The length is 4 inches less than three times the height. The volume is 2240 cubic inches. What are the dimensions of the box?

5. The junior class president and vice president have decided to call all of the junior class to remind them of junior pride week. The president, working alone, can call all of the juniors in six days. The vice president, working along, can call all of the juniors in four days. How long would it take to call of the juniors if they worked together?

6. Suzie can run 2.5 miles per hour faster than Jeff. In the time that it takes Suzie to run 6 miles, Jeff runs 4 miles. Find the speed of each runner.

7. Eva and Emily can clean the entire house in 4 hours. Eva can do it by herself in 6 hours. How long would it take Emily to do it by herself?

8. Sam can paddle a canoe in still water at a speed of 55 meters per minute. If he paddles upstream 135 meters in 3 minutes, what is the speed of the current?

9. The velocity of water flow, in feet per second, from a fire hose nozzle is given by $v(p)=12.1\sqrt{p}$, where p is the nozzle pressure, in pounds per square inch (psi). Find the nozzle pressure if the water flow velocity is 110 feet per second. (Source: Houston Fire Department Continuing Education).

10. The frequency, in hertz, of a violin string can be modeled by the equation $f(t)=49.1\sqrt{t}$, where t is the tension in newtons. What is the amount of tension applied if the frequency of the violin string is 278 hertz?

Using the Structure of Expressions to Solve Equations

Example 5:

Solve $(2x+5)^2-3(2x+5)-40=0$.

$(2x+5)^2-3(2x+5)-40=0$	
$u^2-3u-40=0$ $(u+5)(u-8)=0$ $u+5=0$ or $u-8=0$ $u=-5$ $u=8$	Let $u=2x+5$ and rewrite the equation in terms of *u*. Solve for *u*.
$2x+5=-5$ or $2x+5=8$ $2x=-10$ $2x=3$ $x=-5$ $x=\frac{3}{2}$	Substitute $2x+5$ in for *u* and solve for *x*.

Example 6:

Solve $\frac{1}{(2x-1)^2}+\frac{5}{2x-1}=-6$.

$\frac{1}{(2x-1)^2}+\frac{5}{2x-1}=-6$	
$u^2+5u=-6$ $u^2+5u+6=0$ $(u+3)(u+2)=0$ $u+3=0$ or $u+2=0$ $u=-3$ $u=-2$	Let $u=\frac{1}{2x-1}$ and rewrite the equation in terms of *u*. Solve for *u*.
$\frac{1}{2x-1}=-3$ $\frac{1}{2x-1}=-2$ $1=-3(2x-1)$ $1=-2(2x-1)$ $1=-6x+3$ $1=-4x+2$ $-2=-6x$ $-1=-4x$ $\frac{1}{3}=x$ $\frac{1}{4}=x$	Substitute $\frac{1}{2x-1}$ in for *u* and solve for *x*.

Example 7:
Solve $4x^3 = 8x^2$.

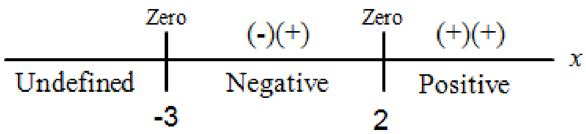
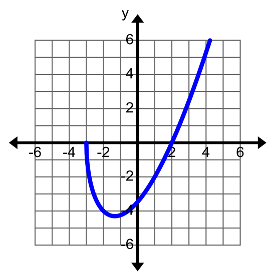
$4x^3 = 8x^2$	
$4x^3 - 8x^2 = 0$ $4x^2(x-2) = 0$	Collect all the terms on one side of the equation and factor the expression.
$4x^2 = 0$ $x-2=0$ $x^2 = 0$ or $x=2$ $x = 0$	Set each factor equal to zero and solve for *x*.

Practice Exercises B

1. $(x+3)^2 - 2(x+3) - 24 = 0$
2. $(x+1)^2 + 8(x+1) + 15 = 0$
3. $3(2-x)^2 + 5(2-x) + 2 = 0$
4. $x + \sqrt{x} = 12$
5. $x + 3\sqrt{x} = 4$
6. $x^{1/2} - 2x^{1/4} + 1 = 0$
7. $\dfrac{1}{(x+2)^2} = \dfrac{1}{x+2} + 2$
8. $\dfrac{1}{(x+5)^2} - \dfrac{4}{x+5} = 12$
9. $\left(\dfrac{x}{x+1}\right)^2 - \dfrac{2x}{x+1} = 8$
10. $x^{2/3} + 9x^{1/3} + 20 = 0$
11. $2x^{2/3} - 5x^{1/3} - 3 = 0$
12. $x^{4/3} - 6x^{2/3} + 9 = 0$
13. $3x^{4/3} + 5x^{2/3} - 2 = 0$
14. $x - 3x\sqrt{x} = 0$
15. $x - x\sqrt{x} = 0$
16. $x^5 + 4x^4 = 21x^3$
17. $x^3 = 16x$
18. $x^5 = 5x^3$
19. $x^3 - 3x^2 - 18x = 0$
20. $x^3 - 3x^2 - 4x + 12 = 0$
21. $x^3 - 3x^2 - x + 3 = 0$

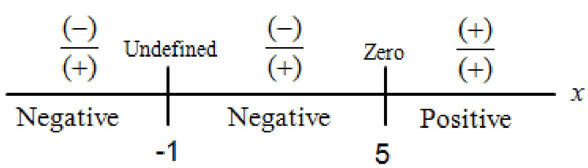
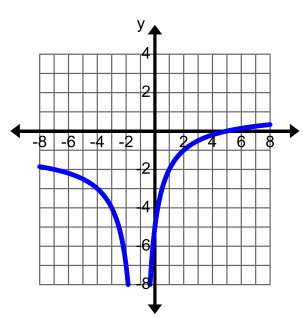
Example 8:

Solve $(x-2)\sqrt{x+3} \geq 0$.

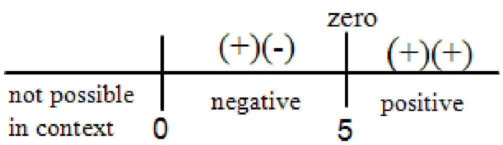
$(x-2)\sqrt{x+3} \geq 0$	The square root makes it so that the expression on the left side of the equation is undefined if $x < -3$. The left side of the equation is zero when $x = -3$ and $x = 2$.
	Create a sign chart, using the zeros, to determine where the expression is positive or equal to zero.
The solution is $\{-3\} \cup [2, \infty)$.	 The graph confirms the solution.

Example 9:

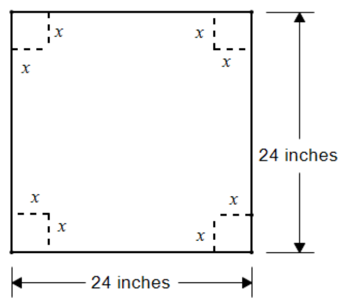
Solve $\dfrac{x-5}{\lvert x+1 \rvert} \leq 0$.

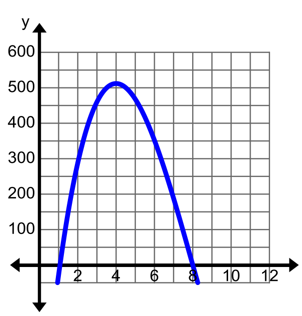
$\dfrac{x-5}{\lvert x+1 \rvert} \leq 0$	The expression on the left is undefined when $x = -1$. The expression is zero when $x = 5$.
	Create a sign chart, using where the function is zero or undefined, to determine where the expression is negative or equal to zero.
The solution is $(-\infty, -1) \cup (-1, 5]$.	 The graph confirms the solution.

Example 10:
The length of a rectangle is five more than the twice the width. If the area is at least 75 square centimeters, what are the possible values for the width?

$A = lw$ $l = 2w + 5$ $w(2w+5) \geq 75$	Write an inequality statement with the known information.
$w(2w+5) \geq 75$ $2w^2 + 5w - 75 \geq 0$ $(2w+15)(w-5) \geq 0$	Gather all the terms on one side of the inequality.
	Create a sign chart.
The width must be greater than or equal to 5 cm or $[5, \infty)$.	

Example 11:
A packaging company is designing a new open-topped box with a volume of at least 512 in^3. The box is to be made from a piece of cardboard measuring 24 inches by 24 inches by cutting identical squares from the corners and turning up the sides. Describe the possible lengths of the sides of the removed squares.



$512 \geq x(24-2x)(24-2x)$	The volume of a rectangular prism is $V = lwh$. The height is x. The width and length are $24-2x$. The length of the side of the square being cut out must be $0 < x < 12$.
$512 \geq x(24-2x)(24-2x)$ $512 \geq 4x^3 - 96x^2 + 576x$ $0 \geq 4x^3 - 96x^2 + 576x - 512$	Expand the equation and get all the terms on the same side so that the expression is compared to zero.
	Use technology to graph the function and find the zeros.
The volume of the box will equal or exceed 512 in^3 if the removed square has a side length on the interval $[1.072, 8]$.	

Practice Exercises C

1. $x^2 + x - 12 \geq 0$
2. $x^2 + 11x + 28 < 0$
3. $x^2 + 3x \geq 4$
4. $4x^3 - 4x > 0$
5. $x^3 + 2x^2 - 15x < 0$
6. $(x+1)(x^2 - 3x + 2) < 0$
7. $x^3 - 6x^2 \leq 7x$
8. $(x+1)(x-3)^2 > 0$
9. $\dfrac{x}{x+3} \geq 0$
10. $\dfrac{x-1}{x^2-4} < 0$
11. $\dfrac{x+2}{x^2-9} \leq 0$
12. $\dfrac{x^2-4}{x^2+4} \geq 0$
13. $x|x-2| > 0$
14. $(2x-1)\sqrt{x+4} < 0$
15. $(3x-4)\sqrt{2x+1} \geq 0$

16. The perimeter of a rectangle is 60 feet. Describe the possible lengths of a side if the area of the rectangle is not to exceed 161 square feet.

17. A diver leaps into the air at 20 feet per second from a diving board that is 12 feet above the water. For how many seconds is the diver at least 10 feet above the water?

18. A projectile is fired straight upward from ground level with an initial velocity of 96 feet per second. During which interval of time will the projectile's height exceed 80 feet?

19. An open box is made from a rectangular piece of cardboard measuring 11 inches by 14 inches by cutting identical squares from the corners and turning up the sides. Describe the possible lengths of the sides of the removed squares if the volume of the open box is not to exceed 132 cubic inches.

20. A calculator company's fixed monthly cost is $25,000 and the cost of producing a single calculator is $75. Describe the company's production level for the month so that the average cost of producing a calculator does not exceed $125.

21. A new drink company is packaging their new cola in 1-liter (1000 cm^3) cylindrical cans. Find the radius of the cans if the cans have a surface area that is less than 750 cm^2.

Solving for a Specified Variable

Example 12: Solve $\frac{1}{c}-\frac{c}{a^2-b^2}=0$ for *c*.

$\frac{1}{c}-\frac{c}{a^2-b^2}=0$	
$\frac{1}{c}-\frac{c}{a^2-b^2}+\frac{c}{a^2-b^2}=0+\frac{c}{a^2-b^2}$ $\frac{1}{c}=\frac{c}{a^2-b^2}$	Add $\frac{c}{a^2-b^2}$ to both sides of the equation.
$1\left(a^2-b^2\right)=c\cdot c$ $a^2-b^2=c^2$	Cross multiply.
$\left(a^2-b^2\right)^{1/2}=\left(c^2\right)^{1/2}$ $\pm\sqrt{a^2-b^2}=c$	Use the properties of exponents to eliminate the square and simplify.

Example 13: Solve $\mu=\sqrt{\frac{3RT}{M}}$ for *R*.

$\mu=\sqrt{\frac{3RT}{M}}$	
$\mu^2=\left(\sqrt{\frac{3RT}{M}}\right)^2$ $\mu^2=\frac{3RT}{M}$	Use the properties of exponents to eliminate the square root.
$M\mu^2=3RT$	Multiply both sides of the equation by *M*.
$\frac{M\mu^2}{3T}=R$	Divide by 3*T*.

Practice Exercises D

Solve each equation for the specified variable.

1. $\frac{q}{m}=\frac{2V}{B^2r^2}$ solve for B

2. $\frac{l}{T^2}=\frac{g}{4\pi^2}$ solve for T

3. $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ solve for c

4. $\frac{r_1}{r_2}=\sqrt{\frac{M_2}{M_1}}$ solve for M_1

5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ solve for y

6. $T=\frac{24(R-r)}{L}$ solve for R

7. $\sqrt{b^2-4ac}=k$ solve for b

8. $4p(y-k)=(x-h)^2$ solve for x

9. $S=\frac{n}{2}(a_1+a_n)$ solve for a_n

10. $a_n=a_1+(n-1)d$ solve for n.

11. $V=\frac{4}{3}\pi r^3$ solve for r

12. $y=\sqrt{a^2-\frac{a^2x^2}{b^2}}$ solve for b.

13. $\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}$ solve for r_2

14. $\frac{V^2}{R^2}=\frac{2g}{R+h}$ solve for h

15. $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ solve for y_2

16. $A=\frac{2Tt+Qq}{2T+Q}$ for Q

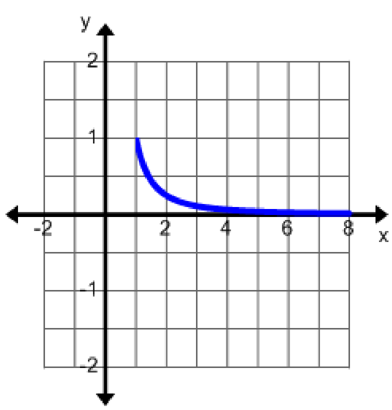
Unit 4 Cluster 1 (A.CED.2 and A.CED.3): Writing and Solving Equations and Inequalities in Two Variables

Cluster 1: Create equations that describe numbers or relationships

- 4.1 Create equations in two or more variables to represent relationships between quantities
- 4.1 Graph equations on coordinate axes with labels and scales
- 4.1 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context

Example 1:

Given the sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$ write and graph the rational equation that models the relationship between the term in the sequence and its value.

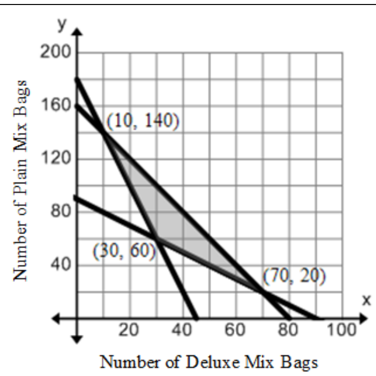
<table>
<tr><td>$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$</td><td></td></tr>
<tr><td>
<table>
<tr><th>Term</th><th>Value</th></tr>
<tr><td>1</td><td>$1 = \frac{1}{1}$</td></tr>
<tr><td>2</td><td>$\frac{1}{2} = \frac{1}{2^2}$</td></tr>
<tr><td>3</td><td>$\frac{1}{9} = \frac{1}{3^2}$</td></tr>
<tr><td>4</td><td>$\frac{1}{16} = \frac{1}{4^2}$</td></tr>
</table>
</td><td>The general term is $\frac{1}{n^2}$.</td></tr>
<tr><td>$f(x) = \frac{1}{x^2}$ when $x \geq 1$ and an integer.


</td><td>Note: this is a graph of the rational equation, not the graph of the sequence.</td></tr>
</table>

Example 2:
All-a-Shirt budgets $6000 to restock 200 shirts. T-shirts sell for $12, polo for $24, and rugby for $36. You need to buy twice as many rugby shirts as polo shirts. If you buy all three types of shirts, how many of each type should you buy?

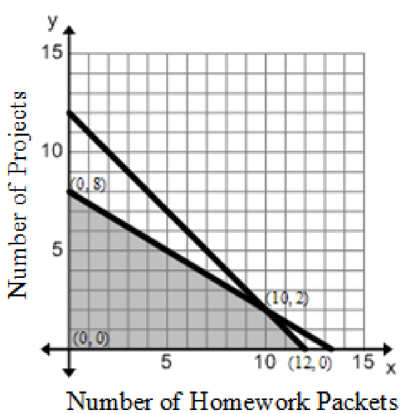
Relationship of polo to rugby: $z = 2y$ Total number of shirts: $x + y + z = 200$ Total cost: $12x + 24y + 36z = 6000$	Write the equations. Let x represent the number of t-shirts, y represent the number of polo shirts, and z represent the number of rugby shirts.
$x + y + 2y = 200$ $x + 3y = 200$ $12x + 24y + 36(2y) = 6000$ $12x + 24y + 72y = 6000$ $12x + 96y = 6000$	Use the relationship of the polo to rugby shirts to rewrite the equations in terms of two variables.
$x + 3y = 200$ $12x + 96y = 6000$ $x = 20,\ y = 60,\ z = 120$	Solve the system of equations using the method of your choice.
You should order 20 t-shirts, 60 polo shirts, and 120 rugby shirts.	

Example 3:
The Sweet Tooth Candy Shoppe is purchasing a candy mix with two types of chocolate: dark chocolate and white chocolate. They need at least 180 pounds of dark chocolate and 90 pounds of white. Their supplier has two mixes for them to buy. The deluxe mix costs $10.00 a bag and has 4 pounds of dark and 1 pound of white. The plain mix costs $5.00 a pound of each. The Sweet Tooth Candy Shoppe can pay at most $800 for the chocolate. How many bags of each can be purchased? Use a graph to help you decide.

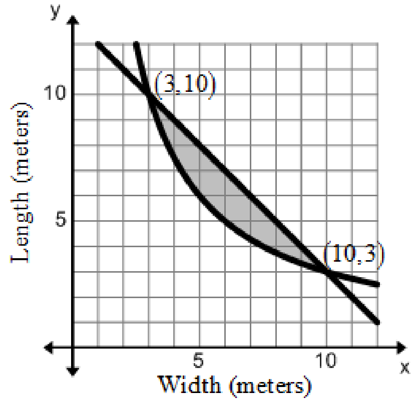
$4x + y \geq 180$ $x \geq 0$ $x + y \geq 90$ $y \geq 0$ $10x + 5y \leq 800$	Let x represent the number of deluxe bags and y represent the number of plain bags. Write all of the constraint equations.
	Graph to find the solution area.

$10(10)+5(140)\le 800$ $800\le 800$ $10(70)+5(20)\le 800$ $800\le 800$ $10(30)+5(60)\le 800$ $600\le 800$	Any point in the shaded area is a solution. However, to minimize the cost check all three of the intersection points in $10x+5y$ to see which one is less than \$800.
Buying 30 bags of the deluxe mixture and 60 bags of the plain mixture will give you the required pounds of chocolate for the least amount of money.	

Example 4:
For his math grade, Carter can do extra homework packets for 70 points each or math projects for 80 points each. He estimates that each homework packet will take 9 hours and each project will take 15 hours and that he will have at most 120 hours to spend. He may turn in a total of no more than 12 packets or projects. How many of each should he complete in order to receive the highest score?

$x+y\le 12$ $x\ge 0$ $9x+15y\le 120$ $y\ge 0$ $70x+80y=\text{maximum}$	Let *x* represent the number of homework packets and *y* represent the number of projects. Write all of the constraint equations.
	Graph to find the solution area.
$70(0)+80(0)=0$ $70(12)+80(0)=840$ $70(10)+80(2)=860$ $70(0)+80(8)=640$	Any point in the shaded region is a solution. To find the highest score, substitute the intersection points into the expression $70x+80y$ to see which one is the greatest.
Completing 10 homework packets and 2 projects will maximize Carter's grade.	

Example 5:
The perimeter of a rectangle is at most 26 meters. Its area is at least 30 square meters. What are the possible dimensions of the rectangle?

$2(x+y) \le 26$ $xy \ge 30$ $x \ge 0$ $y \ge 0$	Write the constraints.
	Graph to find the solution area.

Any point in the shaded region is a solution. For example, $(5,7)$ is a solution because the perimeter would be $2(5+7)=24$ meters which is less than 26 meters. The area would be $5(7)=35$ square meters which is greater than 30 square meters.

Practice Exercises A

For the sequence, write and graph the rational equation that models the relationship between the term in the sequence and its value.

1. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

2. $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \frac{1}{125}, \ldots$

3. $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots$

4. $2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots$

5. $-\frac{1}{2}, 0, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7}, \ldots$

6. $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{12}{7}, \ldots$

Write the constraints for each situation and graph the solution area.

7. A manufacturer produces the following two items: backpacks and messenger bags. Each item requires processing in each of two departments. The cutting department has 60 hours available and finishing department has 42 hours available each week for production. To manufacture a backpack requires 4 hours in cutting and 3 hours in finishing while a messenger bag requires 3 hours in cutting and 2 hours in finishing. Profits on the items are $10 and $7 respectively. If all the bags can be sold, how many of each should be made to maximize profits?

8. Olivia's Orchard consists of 240 acres upon which she wishes to plant red delicious and honey crisp apples. Profit per acre of red delicious is $400. Profit per acre for honey crisp is $300. Furthermore, the total number of hours of labor available during harvest is 3200. Each acre of red delicious requires 20 hours of labor. Each acre of honey crisp requires 10 hours of labor. Determine how the land should be divided to maximize the profits.

9. Kathy owns a car and a moped. She has at most 12 gallons of gasoline to be used between the car and the moped. The car's tank holds at most 18 gallons and the moped's 3 gallons. The mileage for the car is 20 mpg. The mileage for the moped is 100 mpg. How many gallons of gasoline should each vehicle use if Kathy wants to travel as far as possible? What is the maximum number of miles?

10. Cohen is about to take a test that contains short answer questions worth 4 points each and word problems worth 7 points each. Cohen must do at least 5 short answer questions but time restricts doing more than 10. He must do at least 3 word problems but time restricts doing more than 10. Cohen can do no more than 18 questions in total. How many of each type of question must Cohen do in order to maximize his score?

11. Bob's Furniture produces chairs and sofas. The chairs require 20 feet of wood, 1 pound of foam rubber, and 2 square yards of fabric. The sofas require 100 feet of wood, 50 pounds of foam rubber, and 20 square yards of fabric. The company has 1900 feet of wood, 500 pounds of foam rubber, and 240 square yards of fabric. The chairs can be sold for $80 and the sofas for $1,200. How many of each should be produced to maximize the income?

12. The perimeter of the base of a box is no more than 60 inches. If the height is fixed at 8 inches and the volume is at least 1000 cubic inches, what are three possible dimensions of the base of the box?

13. Jen earns $10 per hour for tutoring and $7 per hour as a teacher's aide. Jen must have enough time for studies so she can work no more than 20 hours per week. She must spend at least 3 hours per week tutoring and no more than 8 hours per week tutoring. How many hours a week will she spend tutoring and working as a teacher's aide to maximize the amount she earns?

14. Piper's Paper producers has two factories that produce three types of paper: low, medium, and high grade paper. It supplies no more than 24 tons of low grade, 6 tons of medium grade, and 30 tons of high grade paper. The east factory produces 8 tons of low grade, 1 ton of medium grade, and 2 tons of high grade paper daily and costs $2000 per day to operate. The west factory produces 2 tons of low grade, 1 ton of medium grade, and 8 tons of high grade paper daily and costs $4000 per day to operate. How many days should each factory operate to fill the orders at minimum cost if each factory must produce part of the product?

15. Sally's Scrapbooking prints pages of photographs for albums. A page containing 4 photos will cost $3 while a page containing 6 photos will cost $5. Cyndi can spend no more than $90 for photo pages of her recent vacation and can use no more than 20 pages in her album. What combination of 4-photo pages and 6-photo pages will maximize the number of photos she can display? How many photos can she display?

Building Functions

Unit 4 Cluster 4 (F.BF.1 and F.BF.1c): Combining Functions

Cluster 4: Build a function that models a relationship between two quantities.
4.3 Combine standard function types using arithmetic operations.

Honors
4.3H Compose functions.

Combining functions using arithmetic operations

Let *f* and *g* be any two functions. A new function *h* can be created by performing any of the four basic operations on *f* and *g*.

Operation	**Definition**	**Example:** $f(x)=5x^2+2x$, $g(x)=-3x^2$
Addition	$h(x)=(f+g)(x)$	$h(x)=5x^2+2x+(-3x^2)=2x^2+2x$
Subtraction	$h(x)=(f-g)(x)$	$h(x)=5x^2+2x-(-3x^2)=8x^2+2x$
Multiplication	$h(x)=(fg)(x)$	$h(x)=(5x^2+2x)\cdot(-3x^2)=-15x^4-6x^3$
Division	$h(x)=\left(\dfrac{f}{g}\right)(x)$	$h(x)=\dfrac{5x^2+2x}{-3x^2}=\dfrac{\cancel{x}(5x+2)}{\cancel{x}(-3x)}=\dfrac{5x+2}{-3x}$

The domain of *h* consists of the *x*-values that are in the domains of both *f* and *g*. Additionally, the domain of a quotient does not include *x*-values for which $g(x)=0$.

Example 1:
Let $f(x)=x^2+1$ and $g(x)=\sqrt{x+3}-2$. Find an algebraic expression for $h(x)$ and determine its domain if:

a. $h(x)=(f+g)(x)$

b. $h(x)=(f-g)(x)$

c. $h(x)=(fg)(x)$

d. $h(x)=\left(\dfrac{f}{g}\right)(x)$

$f(x)=x^2+1$ domain is $(-\infty,\infty)$. $g(x)=\sqrt{x+3}-2$ domain is $[-3,\infty)$. The intersection is $[-3,\infty)$.	Determine the domains of $f(x)$ and $g(x)$. Find the intersection of their domains. The intersection will be the domain for the new functions obtained through arithmetic operations. When dividing functions there may be an additional restriction.

a. $h(x)=(f+g)(x)$ $h(x)=(x^2-1)+(\sqrt{x+3}-2)$ $h(x)=x^2+\sqrt{x+3}-3$	Find the sum of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=x^2+\sqrt{x+3}-3$ and its domain is $[-3,\infty)$.	

b. $h(x)=(f-g)(x)$ $h(x)=(x^2-1)-(\sqrt{x+3}-2)$ $h(x)=x^2-\sqrt{x+3}+1$	Find the difference of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=x^2-\sqrt{x+3}+1$ and its domain is $[-3,\infty)$.	

c. $h(x)=(fg)(x)$ $h(x)=(x^2-1)(\sqrt{x+3}-2)$ $h(x)=x^2\sqrt{x+3}-2x^2-\sqrt{x+3}+2$	Find the product of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=x^2\sqrt{x+3}-2x^2-\sqrt{x+3}+2$ and its domain is $[-3,\infty)$.	

d. $h(x)=\left(\frac{f}{g}\right)(x)$ $h(x)=\frac{x^2-1}{\sqrt{x+3}-2}$	Find the quotient of the algebraic expressions representing $f(x)$ and $g(x)$. Notice that $x\neq 1$ or the denominator will be zero.
The new function is $h(x)=\frac{x^2-1}{\sqrt{x+3}-2}$ and its domain is $[-3,1)\cup(1,\infty)$.	

Example 2:

Let $f(x)=x^3$ and $g(x)=\frac{x+1}{x+3}$. Find an algebraic expression for $h(x)$ and determine its domain if:

a. $h(x)=(f+g)(x)$

b. $h(x)=(g-f)(x)$

c. $h(x)=(fg)(x)$

d. $h(x)=\left(\frac{g}{f}\right)(x)$

$f(x)=x^3$ domain is $(-\infty,\infty)$. $g(x)=\dfrac{x+1}{x+3}$ domain is $(-\infty,-3)\cup(-3,\infty)$. The intersection is $(-\infty,-3)\cup(-3,\infty)$.	Determine the domains of $f(x)$ and $g(x)$. Find the intersection of their domains. The intersection will be the domain for the new functions obtained through arithmetic operations. When dividing functions there may be an additional restriction.

a. $h(x)=(f+g)(x)$ $h(x)=(x^3)+\left(\dfrac{x+1}{x+3}\right)$ $h(x)=\dfrac{x^3(x+3)+x+1}{x+3}$ $h(x)=\dfrac{x^4+3x^3+x+1}{x+3}$	Find the sum of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=\dfrac{x^4+3x^3+x+1}{x+3}$ and its domain is $(-\infty,-3)\cup(-3,\infty)$.	

b. $h(x)=(g-f)(x)$ $h(x)=\left(\dfrac{x+1}{x+3}\right)-(x^3)$ $h(x)=\dfrac{x+1-x^3(x+3)}{x+3}$ $h(x)=\dfrac{x+1-x^4-3x^3}{x+3}$	Find the difference of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=\dfrac{-x^4-3x^3+x+1}{x+3}$ and its domain is $(-\infty,-3)\cup(-3,\infty)$.	

c. $h(x)=(fg)(x)$ $h(x)=(x^3)\left(\dfrac{x+1}{x+3}\right)$ $h(x)=\dfrac{x^4+x^3}{x+3}$	Find the product of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $\dfrac{x^4+x^3}{x+3}$ and its domain is $(-\infty,-3)\cup(-3,\infty)$.	

d. $h(x)=\left(\frac{g}{f}\right)(x)$ $h(x)=\frac{\frac{x+1}{x+3}}{x^3}$ $h(x)=\frac{x+1}{x^3(x+3)}$	Find the quotient of the algebraic expressions representing $f(x)$ and $g(x)$. Notice that $x \neq 0$ or the denominator will be zero.
The new function is $h(x)=\frac{x+1}{x^3(x+3)}$ and its domain is $(-\infty,-3)\cup(-3,0)\cup(0,\infty)$.	

Example 3:
Let $f(x)=\cos x$ and $g(x)=2^x+1$. Find an algebraic expression for $h(x)$ and determine its domain if:

a. $h(x)=(f+g)(x)$

b. $h(x)=(f-g)(x)$

c. $h(x)=(fg)(x)$

d. $h(x)=\left(\frac{f}{g}\right)(x)$

$f(x)=\cos x$ domain is $(-\infty,\infty)$. $g(x)=2^x+1$ domain is $(-\infty,\infty)$. The intersection is $(-\infty,\infty)$.	Determine the domains of $f(x)$ and $g(x)$. Find the intersection of their domains. The intersection will be the domain for the new functions obtained through arithmetic operations. When dividing functions there may be an additional restriction.

a. $h(x)=(f+g)(x)$ $h(x)=(\cos x)+(2^x+1)$ $h(x)=\cos x+2^x+1$	Find the sum of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=\cos x+2^x+1$ and its domain is $(-\infty,\infty)$.	

b. $h(x)=(f-g)(x)$ $h(x)=(\cos x)-(2^x+1)$ $h(x)=\cos x-2^x-1$	Find the difference of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=\cos x-2^x-1$ and its domain is $(-\infty,\infty)$.	

c. $h(x)=(fg)(x)$ $h(x)=(\cos x)(2^x+1)$ $h(x)=2^x\cos x+\cos x$	Find the product of the algebraic expressions representing $f(x)$ and $g(x)$.
The new function is $h(x)=2^x\cos x+\cos x$ and its domain is $(-\infty,\infty)$.	

d. $h(x)=\left(\frac{f}{g}\right)(x)$ $h(x)=\frac{\cos x}{2^x+1}$	Find the quotient of the algebraic expressions representing $f(x)$ and $g(x)$. Notice that the denominator will never be zero so there is no restriction.
The new function is $h(x)=\frac{\cos x}{2^x+1}$ and its domain is $(-\infty,\infty)$.	

Practice Exercises A

Find an algebraic expression for $h(x)$ and determine its domain .

1. $f(x)=\sqrt{x-4}+2$ and $g(x)=-3x^2$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{f}{g}\right)(x)$

2. $f(x)=4^{x-2}+1$ and $g(x)=\sqrt{2x}$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{g}{f}\right)(x)$

3. $f(x)=\sin x$ and $g(x)=x^3-3$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{f}{g}\right)(x)$

4. $f(x)=x^2-5x-6$ and $g(x)=-3x+1$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{g}{f}\right)(x)$

5. $f(x)=x^3-x^2$ and $g(x)=x^2-7x+6$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{f}{g}\right)(x)$

6. $f(x)=\dfrac{x-3}{x+5}$ and $g(x)=x+3$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{g}{f}\right)(x)$

7. $f(x)=\cos 3x$ and $g(x)=\sqrt[3]{x+1}$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(f-g)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{f}{g}\right)(x)$

8. $f(x)=\sqrt{x+7}$ and $g(x)=\dfrac{1}{x}$
   a. $h(x)=(f+g)(x)$
   b. $h(x)=(g-f)(x)$
   c. $h(x)=(fg)(x)$
   d. $h(x)=\left(\dfrac{g}{f}\right)(x)$

Evaluating Combined Functions

Example 4:

Let $f(x)=\sqrt{x+2}-2$ and $g(x)=\dfrac{x-4}{x+3}$. Evaluate each of the following:

a. $2f(7)+g(3)$

b. $g(4)-f(-2)$

c. $f(14)\cdot 2g(-1)$

d. $\dfrac{-f(3)}{g(1)}$

a. $2f(7)+g(3)$ $2\left(\sqrt{7+2}-2\right)+\dfrac{3-4}{3+3}$ $2\left(\sqrt{9}-2\right)+\dfrac{-1}{6}$ $2(3-2)-\dfrac{1}{6}$ $2-\dfrac{1}{6}$ $\dfrac{11}{6}$	Substitute $x=7$ in to $f(x)$ and $x=3$ into $g(x)$ and simplify.

b. $g(4)-f(-2)$ $\dfrac{4-4}{4+3}-\left(\sqrt{-2+2}-2\right)$ $\dfrac{0}{7}-\left(\sqrt{0}-2\right)$ $0-(-2)$ $0+2$ 2	Substitute $x=-2$ in to $f(x)$ and $x=4$ into $g(x)$ and simplify.

c. $f(14)\cdot 2g(-1)$ $\left(\sqrt{14+2}-2\right)(2)\left(\dfrac{-1-4}{-1+3}\right)$ $\left(\sqrt{16}-2\right)(2)\left(\dfrac{-5}{2}\right)$ $(4-2)(-5)$ $(2)(-5)$ -10	Substitute $x=-2$ in to $f(x)$ and $x=4$ into $g(x)$ and simplify.

Practice Exercises B

Let $a(x)=x^2-2$, $b(x)=\sqrt{x+1}$, $c(x)=5^{x-3}-2$, $d(x)=2\cos x$, and $f(x)=\dfrac{x}{x-4}$. Evaluate each of the following.

1. $d(\pi)+3f(2)$
2. $-2c(3)+f(1)$
3. $a(-2)+b(3)$
4. $4d(2\pi)-b(8)$
5. $a(-3)-c(4)$
6. $a(0)-4f(0)$
7. $c(2)\cdot b(-1)$
8. $d\left(\dfrac{\pi}{3}\right)\cdot a(-3)$
9. $f(5)\cdot c(4)$
10. $\dfrac{a(-2)}{3b(0)}$
11. $\dfrac{f(3)}{a(1)}$
12. $\dfrac{c(1)}{d(-\pi)}$
13. A company estimates that its cost and revenue can be modeled by the functions $C(x)=-0.75x^2+100x+20,000$ and $R(x)=150x+100$ where x is the number of units produced. The company's profit, P, is modeled by $R(x)-C(x)$. Find the profit equation and determine the profit when 1,000,000 units are produced.
14. Consider the demand equation $p(x)=-\dfrac{1}{15}x+30;\ 0\le x\le 450$ where p represents the price and x the number of units sold. Write an equation for the revenue, R, if the revenue is the price times the number of units sold. What is the revenue if 225 units are sold?
15. The average Cost $\overline{C}$ of manufacturing x computers per day is obtained by dividing the cost function by the number of computers produced that day, x. If the cost function is $C(x)=0.5x^3-34x^2+1213x$, find an equation for the average cost of manufacturing. What is the average cost of producing 100 computers per day?
16. The service committee wants to organize a fund-raising dinner. The cost of renting a facility is \$300 plus \$5 per chair or $C(x)=5x+300$, where x represents the number of people attending the fund-raiser. The committee wants to charge attendees \$30 each or $R(x)=30x$. How many people need to attend the fund-raiser for the event to raise \$1,000?

Composition of Functions (HONORS)

VOCABULARY
Composing one function with another function is applying one function to the result of another function. The notation for composition is $(f \circ g)(x)$ or $f(g(x))$ and $(g \circ f)(x)$ or $g(f(x))$. The inner function is always evaluated in the outer function.
The domain of the composite function is determined by the domain of the inside function and the composite function.

Example 5:

Given $f(x) = x^2 + 2x + 1$ and $g(x) = x + 5$, find $(f \circ g)(x)$ and its domain.

$g(x) = x + 5$ There are no exclusions on the function.	$g(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.
$(f \circ g)(x) = (x+5)^2 + 2(x+5) + 1$ $(f \circ g)(x) = (x^2 + 10x + 25) + (2x + 10) + 1$ $(f \circ g)(x) = x^2 + 12x + 36$	Find the composite. Substitute $g(x)$ into every x in $f(x)$ and simplify.
$(f \circ g)(x) = x^2 + 12x + 36$ There are no exclusions on the function.	Find the domain of $(f \circ g)(x)$.
The composite function is $(f \circ g)(x) = x^2 + 12x + 36$ and its domain is $(-\infty, \infty)$.	

Example 6:

Given $f(x) = \dfrac{1}{x+1}$ and $g(x) = \dfrac{x}{x-2}$, find $(f \circ g)(x)$ and its domain.

$g(x) = \dfrac{x}{x-2}$ $x - 2 = 0$ $x = 2$	$g(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.

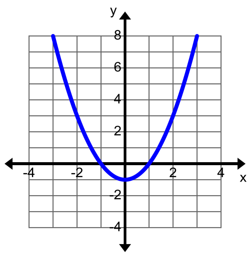
$\left(f\circ g\right)(x)=\dfrac{1}{\dfrac{x}{x-2}+1}$ $\left(f\circ g\right)(x)=\dfrac{1}{\dfrac{x+(x-2)}{x-2}}$ $\left(f\circ g\right)(x)=\dfrac{1}{\dfrac{2x-2}{x-2}}$ $\left(f\circ g\right)(x)=\dfrac{x-2}{2x-2}$	Find the composite. Substitute $g(x)$ into every x in $f(x)$ and simplify.
$\left(f\circ g\right)(x)=\dfrac{x-2}{2x-2}$ $2x-2=0$ $2x=2$ $x=1$	Find the domain of $\left(f\circ g\right)(x)$.
The composite function is $\left(f\circ g\right)(x)=\dfrac{x-2}{2x-2}$ and its domain is $(-\infty,1)\cup(1,2)\cup(2,\infty)$.	

Example 7:
Given $f(x)=x-3$ and $g(x)=2^x$, find $\left(g\circ f\right)(x)$ and its domain.

$f(x)=x-3$ There are no exclusions on the function.	$f(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.
$\left(g\circ f\right)(x)=2^{x-3}$	Find the composite. Substitute $f(x)$ into every x in $g(x)$ and simplify.
$\left(g\circ f\right)(x)=2^{x-3}$ There are no exclusions on the function.	Find the domain of $\left(g\circ f\right)(x)$.
The composite function is $\left(g\circ f\right)(x)=2^{x-3}$ and its domain is $(-\infty,\infty)$.	

Example 8:

Given $f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$, find $(g \circ f)(x)$ and its domain.

$f(x) = x^2 - 1$ There are no exclusions on the function.	$f(x)$ will be the inside function, therefore, determine the values that need to be excluded from its domain.
$(g \circ f)(x) = \sqrt{x^2 - 1}$	Find the composite. Substitute $f(x)$ into every x in $g(x)$ and simplify.
$(g \circ f)(x) = \sqrt{x^2 - 1}$ $x^2 - 1 \geq 0$ $(x-1)(x+1) \geq 0$  The function is positive on the interval $(-\infty, -1) \cup (1, \infty)$.	Find the domain of $(g \circ f)(x)$.
The composite function is $(g \circ f)(x) = \sqrt{x^2 - 1}$ and its domain is $(-\infty, -1) \cup (1, \infty)$.	

Practice Exercises C

Find the indicated composite function and its domain.

1. $f(x) = x - 8$ and $g(x) = \dfrac{1}{x-7}$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

2. $f(x) = 5^{x-4}$ and $g(x) = x^2 - 4$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

3. $f(x) = \sqrt{x-6}$ and $g(x) = x^2 - 3$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

4. $f(x) = |x-5| - 2$ and $g(x) = -2\sin x$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

5. $f(x) = \sqrt[3]{x-2}$ and $g(x) = x^3 + 2$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

6. $f(x) = \dfrac{1}{2x}$ and $g(x) = e^x$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

7. $f(x) = \cos x$ and $g(x) = 4 - x$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

8. $f(x) = \sqrt[3]{x+5}$ and $g(x) = -2|x-4|$
   a. $h(x) = (f \circ g)(x)$
   b. $h(x) = (g \circ f)(x)$
   c. $h(x) = (f \circ f)(x)$
   d. $h(x) = (g \circ g)(x)$

Evaluating Composite Functions

Example 9:
Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Evaluate each of the following:

a. $(f\circ g)(-2)$

b. $(g\circ f)(0)$

c. $(f\circ f)(1)$

d. $(g\circ g)(-7)$

a. $(f\circ g)(-2)=f(g(-2))$ $g(-2)=-2\|-2+4\|+5$ $g(-2)=-2\|2\|+5$ $g(-2)=-4+5$ $g(-2)=1$	Evaluate the inside function $g(x)$ at $x=-2$.
$f(1)=(1-1)^2+3$ $f(1)=0^2+3$ $f(1)=3$	Evaluate the outside function $f(x)$ at the value of $g(-2)$ or $x=1$.
$(f\circ g)(-2)=3$	

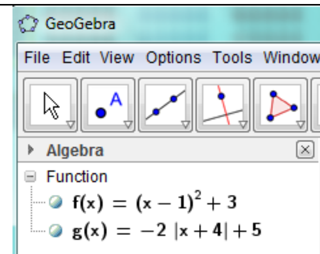
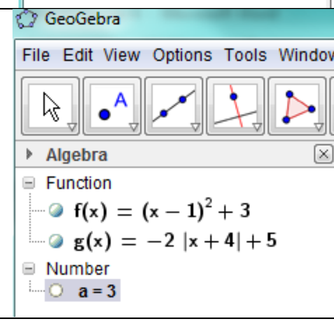
b. $(g\circ f)(0)=g(f(0))$ $f(0)=(0-1)^2+3$ $f(0)=(-1)^2+3$ $f(0)=1+3$ $f(0)=4$	Evaluate the inside function $f(x)$ at $x=0$.
$g(4)=-2\|4+4\|+5$ $g(4)=-2\|8\|+5$ $g(4)=-16+5$ $g(5)=-11$	Evaluate the outside function $g(x)$ at the value of $f(0)$ or $x=4$.
$(g\circ f)(0)=-11$	

c. $(f \circ f)(1) = f(f(1))$ $f(1) = (1-1)^2 + 3$ $f(1) = 0^2 + 3$ $f(1) = 3$	Evaluate the inside function $f(x)$ at $x = 1$.
$f(3) = (3-1)^2 + 3$ $f(3) = 2^2 + 3$ $f(3) = 7$	Evaluate the outside function $f(x)$ at the value of $f(1)$ or $x = 3$.
$(f \circ f)(1) = 7$	

d. $(g \circ g)(-7) = g(g(-7))$ $g(-7) = -2\|-7+4\| + 5$ $g(-7) = -2\|-3\| + 5$ $g(-7) = -6 + 5$ $g(-7) = -1$	Evaluate the inside function $g(x)$ at $x = -7$.
$g(-7) = -2\|-1+4\| + 5$ $g(-7) = -2\|3\| + 5$ $g(-7) = -6 + 5$ $g(-7) = -1$	Evaluate the outside function $g(x)$ at the value of $g(-7)$ or $x = -1$.
$(g \circ g)(-7) = -1$	

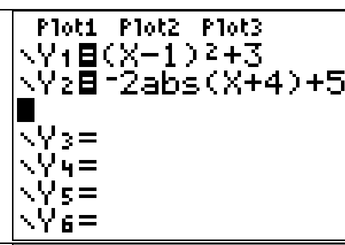
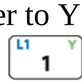
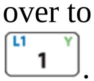
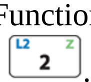
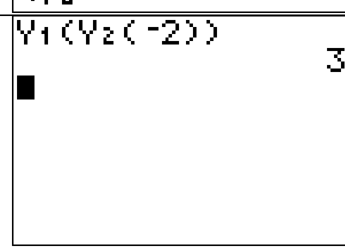
Evaluating Composite Functions using Geogebra

Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Find $(f\circ g)(-2)$.

<table>
<tr><td>With the Algebra window and the Graphics window showing, enter the two functions in the input bar so that they are graphed. They will appear in the Algebra window.</td><td>
</td></tr>
<tr><td>Type $f(g(-2))$ in the input bar then push enter and Geogebra will evaluate the composite function.</td><td>
</td></tr>
</table>

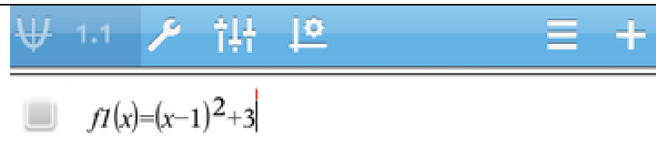
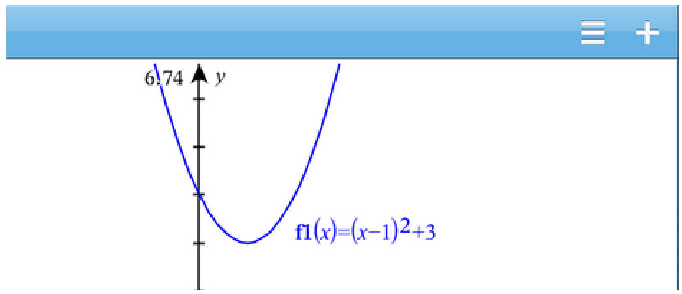
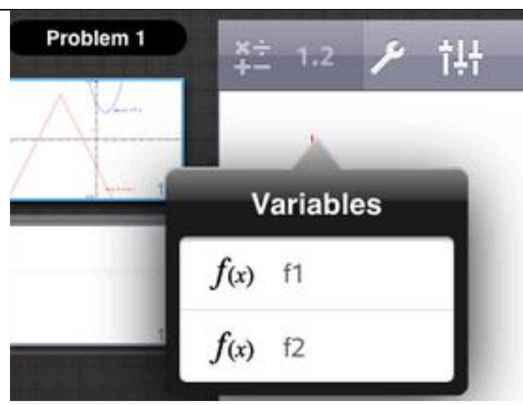
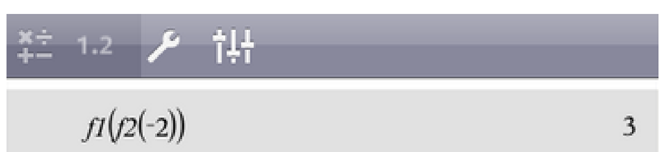
Evaluating Composite Functions using the TI-84 Graphing Calculator

Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Find $(f\circ g)(-2)$.

<table>
<tr><td>Push  and enter $f(x)$ in Y1 and $g(x)$ in Y2. Go back to the home screen by pushing  .</td><td>
</td></tr>
<tr><td>Push , arrow over to Y-VARS and select 1:Function by pushing . $f(x)$ is in Y1 so push . Put a parenthesis  in. Push , arrow over to Y-VARS and select 1:Function by pushing . $g(x)$ is in Y2 so push . Put a parenthesis  in. Enter the value that the composite function is being evaluated at, -2, then close both parentheses. Push  and the calculator will evaluate the composite function.</td><td>
</td></tr>
</table>

Evaluating Composite Functions using the TI-*N*spire CAS App

Let $f(x)=(x-1)^2+3$ and $g(x)=-2|x+4|+5$. Find $(f\circ g)(-2)$.

Select a new document by pushing the + symbol in the upper left corner. Select Graphs so that you graph both functions. Type $f(x)$ in $f1(x)$ and push ENTER to graph it. To enter another equation push the + symbol at the upper right hand corner. Put $g(x)$ in $f2(x)$ and push ENTER to graph it.	  
Add a new window by pushing the + in the upper left had corner. This time select Calculator. Push var and it will show you the functions that are available. Select f1 for $f(x)$. Push var again this time select f2 for $g(x)$. Enter -2 so that it will evaluate the composite at -2. Push ENTER and the answer will appear.	 

Practice Exercises D

Let $f(x)=2x^2+1$, $g(x)=\sqrt{x-2}$, and $h(x)=\dfrac{x-3}{x-5}$. Evaluate each composite function.

1. $(f\circ g)(6)$
2. $(g\circ f)(2)$
3. $(f\circ h)(4)$
4. $(h\circ f)(0)$
5. $(g\circ h)(6)$
6. $(h\circ g)(11)$
7. $(f\circ f)(-1)$
8. $(g\circ g)(18)$
9. $(h\circ h)(-1)$

10. You have a coupon for 15% off a meal at your favorite restaurant. You also have a $10 gift card to the restaurant.

    a. Write a function representing the cost, *x*, of the meal with the coupon.

    b. Write a function representing the cost, *x*, of the meal with the gift card.

    c. Does it save you more money if the coupon is applied first or if the gift card is applied first? How much is the savings?

11. In Utah the general sales tax is 5.95%. If you have a coupon for 10% off your entire bill, is it better to have the sales tax applied before or after the 10% off coupon?

12. A balloon's radius can be modeled by the equation $r(t)=0.05t+4$, where *t* is the time in seconds and *r* is measured in centimeters. The volume of a sphere is $V(r)=\dfrac{4}{3}\pi r^3$. Write the formula for $(V\circ r)(t)$. Find the volume of the balloon at 30 seconds.

Unit 4 Cluster 5 (F.BF.4): Inverses

Cluster 5: Build new functions from existing functions
  4.5 Find inverse functions
Honors
  4.5 Verify by composition that one function is the inverse of another.
  4.5 Read values of an inverse function from a graph or a table, given that the function has an inverse.
  4.5 Produce an invertible function from a non-invertible function by restricting the domain.

VOCABULARY

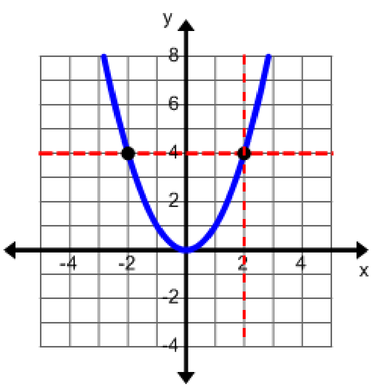
If no vertical line intersects the graph of a function *f* more than once, then *f* is a function. This is called the **vertical line test**. If no horizontal line intersects the graph of a function *f* more than once, then the inverse of *f* is itself a function. This is called the **horizontal line test**.

The **inverse of a function** is formed when the independent variable is exchanged with the dependent variable in a given relation. (Switch the *x* and *y* with each other.) A function takes a starting value, performs some operation on this value, and creates an output answer. The inverse of a function takes the output answer, performs some operation on it, and arrives back at the original function's starting value. Inverses are indicated by the notation f^{-1}.

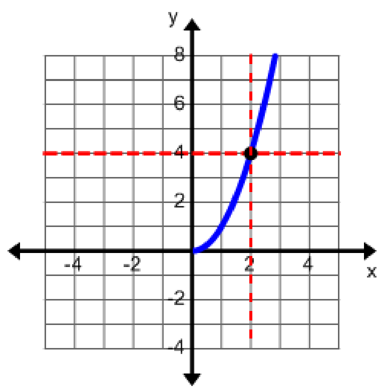
A function is a **one-to-one function** if and only if each second element corresponds to one and only one first element.

In order for the inverse of a function to be a function, the original function must be a one-to-one function and meets the criteria for the vertical and horizontal line tests.

Not all functions meet the criteria to have an inverse which is also a function. However, if the **domain is restricted**, or in other words only part of the domain is used, then the inverse will be a function.



This example is not one-to-one. It is a function because the vertical line intersects the graph only once. However, the horizontal line intersects the graph twice. There is an inverse to this example, but the inverse will not be a function.



This example is one-to-one. It is a function because the vertical and horizontal lines intersect the graph only once. The inverse will be a function.

Example 1:

Find the inverse of $f(x)=5x+7$.

$y=5x+7$	$f(x)$ is the same as *y* so replace $f(x)$ with *y*.
$x=5y+7$	Substitute *x* with *y* and *y* with *x*.
$x=5y+7$ $x-7=5y$ $\frac{x-7}{5}=y$	Solve for *y*.
$f^{-1}(x)=\frac{x-7}{5}$	This is the inverse of $f(x)$.

Example 2:

Find the inverse of $f(x)=x^2-6x+13$ when $x\le 3$.

$y=x^2-6x+13$	$f(x)$ is the same as *y* so replace $f(x)$ with *y*.
$x=y^2-6y+13$	Substitute *x* with *y* and *y* with *x*.
$x=\left(y^2-6y+9\right)+13-9$ $x=(y-3)^2+4$ $x-4=(y-3)^2$ $\pm\sqrt{x-4}=y-3$ $\pm\sqrt{x-4}+3=y$	Complete the square in order to solve for *y*.
$\pm\sqrt{x-4}+3=y$ $-\sqrt{x-4}+3=y$	The domain of the original function is the range of the inverse function. Therefore, we select the negative root.
$f^{-1}(x)=-\sqrt{x-4}+3$	This is the inverse of $f(x)$.

Example 3:

Find the inverse of $f\left(x\right)=\dfrac{4x+3}{3x-1}$.

$y=\dfrac{4x+3}{3x-1}$	$f\left(x\right)$ is the same as y so replace $f\left(x\right)$ with y.
$x=\dfrac{4y+3}{3y-1}$ $x\left(3y-1\right)=4y+3$ $3xy-x=4y+3$ $3xy-4y=x+3$ $y\left(3x-4\right)=x+3$ $y=\dfrac{x+3}{3x-4}$	Substitute each x with y and y with x.
$f^{-1}\left(x\right)=\dfrac{x+3}{3x-4}$	This is the inverse of $f\left(x\right)$.

Practice Exercises A

Find the inverse of each function.

1. $f\left(x\right)=-6x+8$
2. $f\left(x\right)=-\dfrac{1}{2}x-2$
3. $f\left(x\right)=3x-5$
4. $f\left(x\right)=-\dfrac{1}{4}x^2+3,\ x\le 0$
5. $f\left(x\right)=x^2-4x-12,\ x\ge 2$
6. $f\left(x\right)=2x^2+12x+14,\ x\ge -3$
7. $f\left(x\right)=\sqrt{x+4}$
8. $f\left(x\right)=\sqrt{2x-5}+4$
9. $f\left(x\right)=-2\sqrt{3-x}$
10. $f\left(x\right)=\sqrt{x+1}-3$
11. $f\left(x\right)=\dfrac{3x+5}{x-1}$
12. $f\left(x\right)=\dfrac{-2x+1}{5x-6}$
13. $f\left(x\right)=\dfrac{7x-6}{3x+2}$
14. $f\left(x\right)=\dfrac{4x-3}{x+4}$
15. $f\left(x\right)=\dfrac{1}{2}x^3-3$
16. $f\left(x\right)=-3x^3+7$
17. $f\left(x\right)=\left(x-2\right)^3+5$
18. $f\left(x\right)=\sqrt[3]{x+2}-3$
19. $f\left(x\right)=-2\sqrt[3]{x-5}+7$
20. $f\left(x\right)=\dfrac{1}{2}\sqrt[3]{4-x}+1$
21. $f\left(x\right)=\sqrt[3]{x-1}-4$

Composition of Inverse Functions
If *f* and f^{-1} are inverse functions, then $\left(f^{-1} \circ f\right)(x) = x$ and $\left(f \circ f^{-1}\right)(x) = x$ for all *x* in the domains of *f* and f^{-1} respectively.

Example 4:

Verify that $f(x) = 3x - 2$ and $g(x) = \dfrac{x+2}{3}$ are inverses of each other.

$(f \circ g)(x) = 3\left(\dfrac{x+2}{3}\right) - 2$ $(f \circ g)(x) = x + 2 - 2$ $(f \circ g)(x) = x$	Find $(f \circ g)(x)$.
$(g \circ f)(x) = \dfrac{(3x-2)+2}{3}$ $(g \circ f)(x) = \dfrac{3x}{3}$ $(g \circ f)(x) = x$	Find $(g \circ f)(x)$.
Both $(f \circ g)(x)$ and $(g \circ f)(x)$ are equal to *x*, therefore, $f(x)$ and $g(x)$ are inverses of each other.	

Practice Exercises B

Verify that *f* and *g* are inverses are of each other.

1. $f(x) = \dfrac{x+3}{4}$ and $g(x) = 4x - 3$
2. $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$
3. $f(x) = \dfrac{1}{2x}$ and $g(x) = \dfrac{1}{2x}$
4. $f(x) = \sqrt{x+2} - 1$ and $g(x) = (x+1)^2 - 2$, $x \geq -1$
5. $f(x) = \dfrac{1}{2}x - 4$ and $g(x) = 2x + 8$
6. $f(x) = (x-3)^3 - 2$ and $g(x) = \sqrt[3]{x+2} + 3$
7. $f(x) = \dfrac{5}{x+1}$ and $g(x) = \dfrac{5-x}{x}$
8. $f(x) = \dfrac{1}{2}x^2 - 5$, $x \geq 0$ and $g(x) = \sqrt{2x+10}$
9. $f(x) = 2x + 4$ and $g(x) = \dfrac{1}{2}x - 2$
10. $f(x) = \dfrac{x+3}{x-2}$ and $g(x) = \dfrac{2x+3}{x-1}$

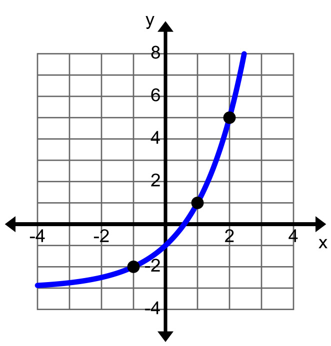
Example 5:
Use the table below to write the table for the inverse function.

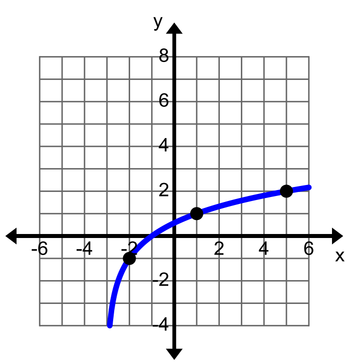
x	$f(x)=x^3-4x+1$
-7	-314
-6	-191
-5	-104
-4	-47
-3	-14
-2	1
-1	4

x	$f^{-1}(x)$
-314	-7
-191	-6
-104	-5
-47	-4
-14	-3
1	-2
4	-1

The domain of the original function becomes the range of the inverse function and the range of the original function becomes the domain of the inverse function.

Example 6:
Use the graph below to draw the graph of the inverse function.





The domain of the original function becomes the range of the inverse function and the range of the original function becomes the domain of the inverse function.

Practice Exercises C

Use the table to write the table for the inverse function.

1.

x	$f(x)$
-2	0.5
-1	1.5
0	4.5
1	13.5
2	40.5

2.

x	$f(x)$
2	3
3	4
5	5
9	6
17	7

3.

x	$f(x)$
5	1
6	3
9	4
14	5
21	6

4.

x	$f(x)$
0	4
2	2
4	-4
6	-14
8	-28

5.

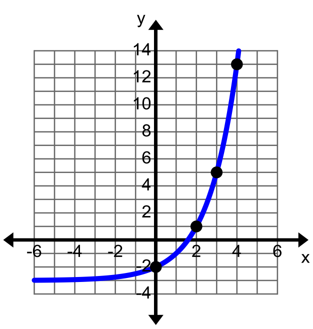
x	$f(x)$
-17	1.7
-12	1.6
-9	1.5
-7	1.4
-3	1

6.

x	$f(x)$
-4	0
-3.5	1.125
-3	21
-2.5	24.375
-2	24

Use the graph to draw the graph of the inverse function.

7.



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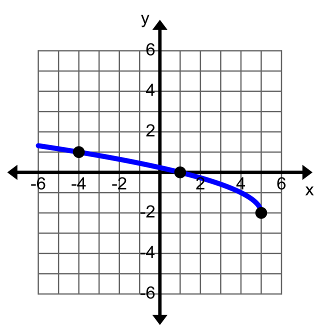

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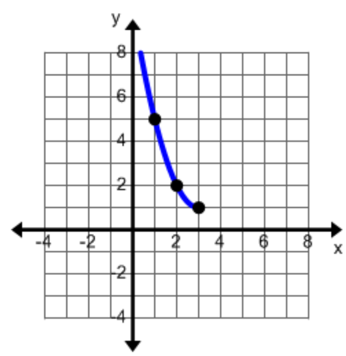

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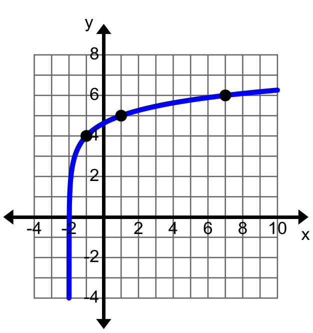

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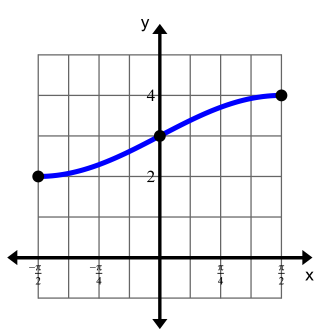

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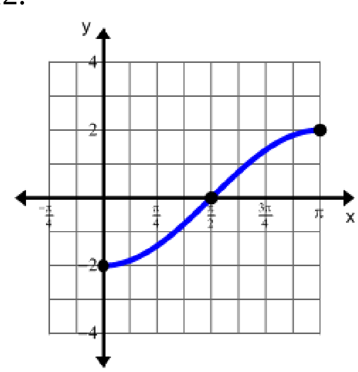
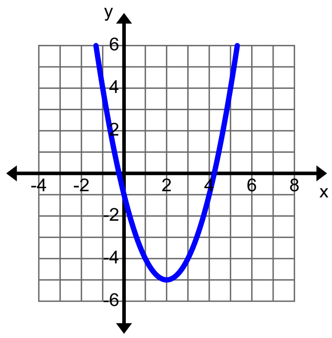


Example 7:

Given $f(x)=x^2-4x-1$, find a suitable domain to make this function an invertible function.

	Graph the function.
Increasing $(2,\infty)$ Decreasing $(-\infty,2)$	Determine the intervals where the function is increasing and decreasing.
Restrict the domain to $x\ge 2$ or $x\le 2$.	Choose an interval where the function is monotonic (meaning solely increasing or solely decreasing).

Practice Exercises D

For each function, find a suitable domain to make the function an invertible function.

1. $f(x)=2x^2-3$
2. $f(x)=-(x+2)^2$
3. $f(x)=(x+5)^2+4$
4. $f(x)=x^2+2x-3$
5. $f(x)=x^2+12x+32$
6. $f(x)=2x^2-12x+15$
7. $f(x)=|x-3|+4$
8. $f(x)=-2|x|+6$
9. $f(x)=|2-x|-5$
10. $f(x)=-\frac{1}{2}|x+4|$
11. $f(x)=(x-7)^4+8$
12. $f(x)=(x+5)^4-3$

Logarithms

Unit 4 Cluster 6 (F.LE.4 and F.BF.5): Logarithms

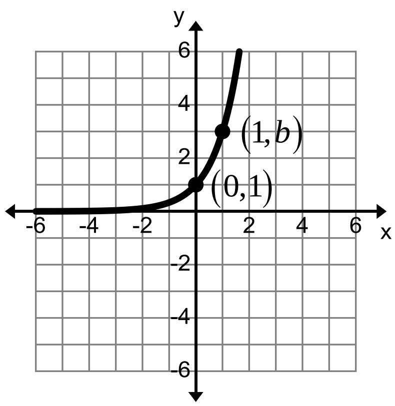
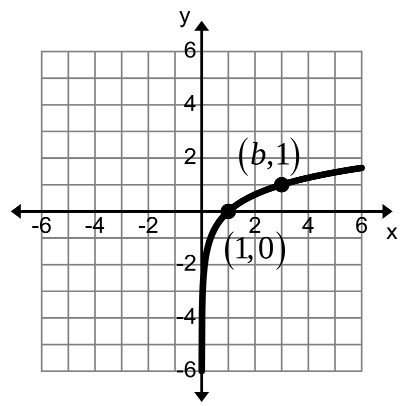
Cluster 6: Logarithms

4.6 For exponential models, express as a logarithm the solution to a $b^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10 or *e*.

4.6 Evaluate the logarithm using technology.

4.6 Understand the inverse relationship between exponents and logarithms.

4.6 Use the relationship between exponentials and logarithms to solve problems involving logarithms and exponents.

$f(x) = b^x,\ b \neq 0,\ b \neq 1$	$f(x) = \log_b x,\ b \neq 0,\ b \neq 1$
	
Domain: $(-\infty, \infty)$ Range: $(0, \infty)$ Horizontal Asymptote: $y = 0$ Intercept: $(0,1)$ End Behavior: $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to -\infty} f(x) = 0$	Domain: $(0, \infty)$ Range: $(-\infty, \infty)$ Vertical Asymptote: $x = 0$ Intercept: $(1, 0)$ End Behavior: $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to 0^+} f(x) = -\infty$

Exponential and logarithmic functions are inverses of each other. Two of the most widely used logarithms are the common log, which is base 10, and is written as $\log_{10} x = \log x$ and the natural log, which is base *e*, and is written as $\log_e x = \ln x$.

Definition of a Logarithm
$\log_b x = c$ if and only if $b^c = x$ $\ln x = c$ if and only if $e^c = x$

Example 1:
Rewrite each of the following in exponential form.

a. $\log_4 64 = 3$ b. $\log_5 \frac{1}{25} = -2$ c. $\log_{65} 1 = 0$

a. $4^3 = 64$	The base is 4 and the exponent is 3.
b. $5^{-2} = \frac{1}{25}$	The base is 5 and the exponent is -2.
c. $65^0 = 1$	The base is 65 and exponent is 0.

Example 2:
Rewrite each of the following in logarithmic form.

a. $3^4 = 81$ b. $10^{-2} = \frac{1}{100}$ c. $6^1 = 6$

a. $\log_3 81 = 4$	The base is 3 and the exponent is 4.
b. $\log_{10} \frac{1}{100} = -2$	The base is 10 and the exponent is -2.
c. $\log_6 6 = 1$	The base is 6 and exponent is 1.

Basic Properties of Logarithms
where $b > 0$, $b \neq 1$, $x > 0$, and c is any real number

1. $\log_b 1 = 0$	1. $\ln 1 = 0$
2. $\log_b b = 1$	2. $\ln e = 1$
3. $\log_b b^c = c$	3. $\ln e^c = c$
4. $b^{\log_b x} = x$	4. $e^{\ln x} = x$

Example 3:
Use the properties of logarithms to evaluate the expression without a calculator.

a. $\log 10^{-4}$ b. $e^{\ln 6}$ c. $\log_3 1$ d. $\log_{50} 50$

a. $\log 10^{-4} = -4$	Use of basic property 3.
b. $e^{\ln 6} = 6$	Use of basic property 4.
c. $\log_3 1 = 0$	Use of basic property 1.
d. $\log_{50} 50 = 1$	Use of basic property 2.

Practice Exercises A

Rewrite each of the equations in exponential form.

1. $\log_4 1 = 0$
2. $\log_2 8 = 3$
3. $\log_9 \dfrac{1}{81} = -2$
4. $\log_3 243 = 5$
5. $\log_7 \dfrac{1}{343} = -3$
6. $\log_6 216 = 3$

Rewrite each of the equations in logarithmic form.

7. $10^{-3} = \dfrac{1}{1000}$
8. $5^4 = 625$
9. $6^{-1} = \dfrac{1}{6}$
10. $9^3 = 729$
11. $7^1 = 7$
12. $3^3 = 27$

Use the properties of logarithms to evaluate the expression without a calculator.

13. $\log_8 8$
14. $\log_9 9^3$
15. $\ln 1$
16. $4^{\log_4(2x)}$
17. $\log_5 5^{3-x}$
18. $\log_6 1$
19. $\log_9 9$
20. $e^{\ln x^2}$
21. $\ln e^{10x+5}$

The Principle of Exponential Equality

For any real number *b*, where $b \neq -1,\ 0,$ or 1, $b^{x_1} = b^{x_2}$ is equivalent to $x_1 = x_2$. In other words, powers of the same base are equal if and only if the exponents are equal.

Example 4:
Solve the following.

a. $\log_2 8 = x$ b. $\log_7 x = 2$ c. $\log_x 125 = 3$

a. $2^x = 8$ $2^x = 2^3$ $x = 3$	Rewrite in exponential form and solve for *x*.
b. $7^2 = x$ $49 = x$	Rewrite in exponential form and solve for *x*.
c. $x^3 = 125$ $x^3 = 5^3$ $x = 5$	Rewrite in exponential form and solve for *x*.

Example 5:
Solve the following.

a. $27^2 = 9^{x+1}$ b. $\log_4(3x-2) = 2$

a. $27^2 = 9^{x+1}$ $(3^3)^2 = (3^2)^{x+1}$ $3^6 = 3^{2x+2}$ $6 = 2x+2$ $4 = 2x$ $2 = x$	Change the bases on both sides of the equation so that they are the same base. Rewrite using exponent rules. Simplify using exponent rules. Solve using the principle of exponential equality.
b. $\log_4(3x-2) = 2$ $4^2 = 3x-2$ $16 = 3x-2$ $18 = 3x$ $6 = x$	Rewrite in exponential form and solve for *x*.

Practice Exercises B

Solve the following equations.

1. $\log_7 x = 3$	2. $\log_5 625 = x$	3. $\log_x 10,000 = 4$
4. $\log_x \frac{1}{225} = -2$	5. $\log_9 x = 0$	6. $\log_3 \frac{1}{243} = x$
7. $4^3 = 8^{3-x}$	8. $16^{x+1} = 8^{x+3}$	9. $\left(\frac{1}{9}\right)^{2-x} = 27^2$
10. $\log_7(10x+3) = 3$	11. $\log_2(17x-2) = 5$	12. $\log_4(3x-5) = 3$

For each of the following rules, $b \neq 1$, x, y, and c are real numbers.

Product Rule

$\log_b(xy) = \log_b x + \log_b y$ $\qquad$ $\ln(xy) = \ln x + \ln y$

Quotient Rule

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$ $\qquad$ $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

Power Rule

$\log_b(x)^c = c\log_b x$ $\qquad$ $\ln(x)^c = c\ln x$

Example 6:
Expand the following expressions.

a. $\log \frac{a^4 b}{c^5}$ $\qquad$ b. $\ln\sqrt{m^3 n}$ $\qquad$ c. $\log \frac{2w^4 h^3}{a^2 b^5}$

a.	
$\log \frac{a^4 b}{c^5}$	
$\log a^4 + \log b - \log c^5$	Use the product and quotient rules to rewrite the expression.
$4\log a + \log b - 5\log c$	Use the power rule to rewrite the expression.

b. $\ln\sqrt{m^3 n}$ $\frac{1}{2}\ln m^3 n$ $\frac{1}{2}\left(\ln m^3 + \ln n\right)$ $\frac{1}{2}\left(3\ln m + \ln n\right)$ $\frac{3}{2}\ln m + \frac{1}{2}\ln n$	Use the power rule to rewrite the expression. Use the product rule to rewrite the epxression. Use the power rule to rewrite the expression. Use the distributive property.
c. $\log\frac{2w^4h^3}{a^2b^5}$ $\log 2 + \log w^4 + \log h^3 - \left(\log a^2 + \log b^5\right)$ $\log 2 + 4\log w + 3\log h - \left(2\log a + 5\log b\right)$ $\log 2 + 4\log w + 3\log h - 2\log a - 5\log b$	Use the product and the quotient rules to rewrite the expression. Use the power rule to rewrite the expression. Use the distributive property.

Note: When using the quotient rule, all terms in the denominator will be subtracted.

Example 7:
Condense the following expressions.

a. $\ln\left(x+1\right) - 3\ln\left(x-2\right)$ b. $\log 3 + 4\log a - \frac{2}{3}\log b$ c. $4\ln a - 3\ln b + 7\ln c - 5\ln\left(d+1\right)$

a. $\ln\left(x+1\right) - 3\ln\left(x-2\right)$ $\ln\left(x+1\right) - \ln\left(x-2\right)^3$ $\ln\frac{x+1}{\left(x-2\right)^3}$	Use the power rule to rewrite the expression. Use the quotient rule to rewrite the expression.

b. $\log 3 + 4\log a - \frac{2}{3}\log b$ $\log 3 + \log a^4 - \log b^{2/3}$ $\log\frac{3a^4}{\sqrt[3]{b^2}}$	Use the power rule to rewrite the expression. Use the product and quotient rules to rewrite the expression

c. $4\ln a - 3\ln b + 7\ln c - 5\ln(d+1)$ $\ln a^4 - \ln b^3 + \ln c^7 - \ln(d+1)^5$ $\ln \dfrac{a^4 c^7}{b^3 (d+1)^5}$	Use the power rule to rewrite the expression. Use the product and quotient rules to rewrite the expression.

Practice Exercises C

Expand the following expressions.

1. $\log_4 x^5 y^7$
2. $\log_7 49xyz$
3. $\log \dfrac{a^2 b^3}{c^4}$
4. $\log\left[(2x+1)(x+7)\right]$
5. $\log_8 8\sqrt{3a^5}$
6. $\log\left(\dfrac{\sqrt{x}\,y^3}{z^3}\right)$
7. $\log_3 \dfrac{27(x-3)}{x^2 y^5}$
8. $\log_5 \sqrt[3]{\dfrac{x^2 y}{25}}$
9. $\ln \dfrac{x^3\sqrt{x^2+1}}{(x+1)^4}$

Condense the following expressions.

10. $\ln x + \ln 7$
11. $\log_2 96 - \log_2 3$
12. $\log(2x+5) - \log(x-3)$
13. $4\ln x + 7\ln y - 3\ln z$
14. $5\log_6 x + 2\log_6 y - \dfrac{2}{3}\log_6 z$
15. $\dfrac{1}{2}\log_3 x + \dfrac{1}{2}\log_3 z - \dfrac{3}{2}\log_3 y$
16. $\log x + \log 7 + \log(x^2-1) - \log(x+1)$
17. $\ln(x-2) - \ln(x^2-4) - 3\ln x$

Change of Base Formula for Logarithms

Most calculators only have $\log x$ and $\ln x$. In order to evaluate logarithms with a different base, you will need the change of base formula.

$$\log_b x = \frac{\log x}{\log b},\ b \neq 1 \qquad \text{or} \qquad \log_b x = \frac{\ln x}{\ln b},\ b \neq 1$$

Example 8:
Find an approximation for the following expressions.

a. $\ln 7$ b. $\log 0.15$ c. $\log_4 17$ d. $\log_{52} 26$

a. $\ln 7 \approx 1.176$	Use the calculator.
b. $\log 0.15 \approx -0.824$	Use the calculator.
c. $\log_4 17 = \frac{\log 17}{\log 4} \approx 2.044$ $\log_4 17 = \frac{\ln 17}{\ln 4} \approx 2.044$	Use the change of base formula and your calculator.
d. $\log_{52} 26 = \frac{\log 26}{\log 52} \approx 0.825$ $\log_{52} 26 = \frac{\ln 26}{\ln 52} \approx 0.825$	Use the change of base formula and your calculator.

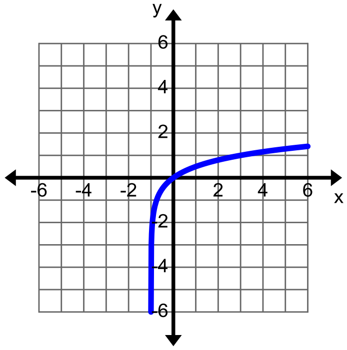
Practice Exercises D

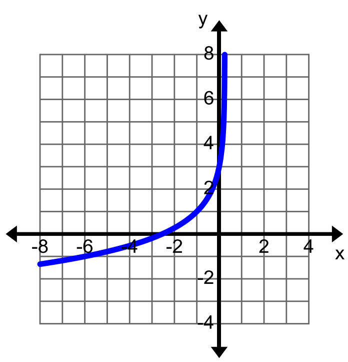
Find an approximation for the following expressions.

1. $\ln 0.5$
2. $\log 3.25$
3. $\ln 56$
4. $\log 20$
5. $\log 0.125$
6. $\log_4 200$
7. $\log_6 7780$
8. $\log_5 10$
9. $\log_3 12$
10. $\log_7 47$
11. $\log_{14} 100$
12. $\log_9 1000$

Example 9:
Find the domain of the function then graph it.

a. $f(x) = \log_4(x+1)$ b. $f(x) = -2\log_5(4x+1)+3$

a. $x+1>0$ $x>-1$ The domain is $(-1,\infty)$.	The domain of a logarithmic function has to be greater than zero.
	Use the change of base formula to enter the function. $y = \dfrac{\log(x+1)}{\log 4}$ or $y = \dfrac{\ln(x+1)}{\ln 4}$

b. $1-4x>0$ $-4x>-1$ $x<\dfrac{1}{4}$ The domain is $\left(-\infty,\dfrac{1}{4}\right)$.	The domain of a logarithmic function has to be greater than zero.
	Use the change of base formula to enter the function. $y = -2\dfrac{\log(1-4x)}{\log 5}+3$ or $y = -2\dfrac{\ln(1-4x)}{\ln 5}+3$

Practice Exercises E

Find the domain of the function and then graph it.

1. $f(x)=\log_5(x+4)$	2. $f(x)=\log_3(x+6)$	3. $f(x)=\log_4(2-x)$
4. $f(x)=\log_2(7-x)$	5. $f(x)=\log_6(2x+1)$	6. $f(x)=\log_7(4-3x)$
7. $f(x)=\log_2(x-5)+2$	8. $f(x)=-4\log_8(-2x)+7$	9. $f(x)=\log_5(x-8)-1$
10. $f(x)=3\log_4(5-3x)-2$	11. $f(x)=\log_3(5x-6)-4$	12. $f(x)=\log_6(8x-5)+6$

The Principle of Logarithmic Equality

For any logarithmic base, *b*, and for any $x>0$ and $y>0$, $x=y$ is equivalent to $\log_b x=\log_b y$. In other words, two expressions are equal if and only if the logarithms of those expressions are equal.

Example 10:
Solve each equation.

a. $\log_4(x-5)=-1$

b. $50e^{0.035x}=200$

c. $\ln(x-3)+\ln(x+4)=3\ln(2)$

d. $25^{2x+1}=144$

a. $$\begin{aligned}\log_4(x-5)&=-1\\ 4^{-1}&=x-5\\ \frac{1}{4}&=x-5\\ \frac{21}{4}&=x\end{aligned}$$	Write the equation in exponential form. Solve for *x*.
$$\begin{aligned}\log_4\left(\frac{21}{4}-5\right)&\stackrel{?}{=}-1\\ \log_4\left(\frac{1}{4}\right)&\stackrel{?}{=}-1\\ -1&=-1\end{aligned}$$	Check your answer in the original equation.

b. $50e^{0.035x} = 200$ $e^{0.035x} = 4$ $\ln\left(e^{0.035x}\right) = \ln 4$ $0.035x = \ln 4$ $x = \dfrac{\ln 4}{0.035} \approx 39.608$	Isolate the exponential term. Find the natural logarithm of both sides. Use the property $\ln e^c = c$ to eliminate the base. Solve for *x*.

c. $\ln(x-3)+\ln(x+4) = 3\ln(2)$ $\ln((x-3)(x+4)) = \ln 2^3$ $\ln\left(x^2+x-12\right) = \ln 8$ $x^2+x-12 = 8$ $x^2+x-20 = 0$ $(x+5)(x-4) = 0$ $x+5=0 \qquad x-4=0$ $x=-5 \qquad x=4$	Use the product rule and the power rule to rewrite both sides of the equation. Expand both sides of the equation. Use the principal of logarithmic equality to eliminate the logarithm. Solve for *x*.
$\ln(-5-3)+\ln(-5+4) \stackrel{?}{=} 3\ln(2)$ $\ln(-8)+\ln(-1) \stackrel{?}{=} 3\ln(2)$ $x=-5$ is an extraneous solution. $\ln(4-3)+\ln(4+4) \stackrel{?}{=} 3\ln(2)$ $\ln(1)+\ln(8) \stackrel{?}{=} 3\ln(2)$ $0+\ln(8) \stackrel{?}{=} 3\ln(2)$ $\ln\left(2^3\right) \stackrel{?}{=} 3\ln(2)$ $3\ln(2) = 3\ln(2)$ $x=4$ is a solution.	Check both answers in the original equation. The $\ln(-8)$ and $\ln(-1)$ are undefined.

d. $$25^{2x+1}=144$$	
$$\ln\left(25^{2x+1}\right)=\ln 144$$	Find the natural log of both sides.
$$(2x+1)\ln 25=\ln 144$$	Use the power rule to rewrite the expression.
$$2x\ln 25+\ln 25=\ln 144$$	Use the distributive property.
$$2x\ln 25=\ln 144-\ln 25$$ $$x=\frac{\ln 144-\ln 25}{2\ln 25}\approx 0.272$$	Solve for *x*.

Practice Exercises F

Solve each equation.

1. $\log_3(2x-1)=3$
2. $\log_7(3x-11)=\log_7(x-3)$
3. $\log_6 x+\log_6 3=2$
4. $2\log_3(x+4)-\log_3 9=2$
5. $3\log_4(x-2)+\log_4 16=5$
6. $\log x+\log(x-21)=2$
7. $\log_6 x+\log_6(x-9)=2$
8. $\log_2(x+2)+\log_2(x+4)=3$
9. $\log_3(x+6)=1-\log_3(x+4)$
10. $3^x=25$
11. $2^{-x}=1.5$
12. $5^{x+3}=30$
13. $30e^{0.6x}=240$
14. $7e^{2x}=63$
15. $3\cdot 4^{2x-1}=42$
16. $4^{5-x}-2=13$
17. $5^{4x-7}-3=10$
18. $6^{3x-4}-7=65$

Example 11:
Find the inverse of each function.

a. $f(x)=\ln(x+2)-3$

b. $f(x)=\log_3(2x+1)+5$

c. $f(x)=5^{x-6}+1$

d. $f(x)=5\cdot 2^{3-x}-4$

a.	
$y=\ln(x+2)-3$	Substitute each x with y and y with x.
$x=\ln(y+2)-3$	Isolate the logarithmic term.
$x+3=\ln(y+2)$	
$e^{x+3}=e^{\ln(y+2)}$	Use the property $e^{\ln x}=x$ to eliminate the logarithm.
$e^{x+3}=y+2$	
$e^{x+3}-2=y$	Solve for y.
$e^{x+3}-2=f^{-1}(x)$	

b.	
$y=\log_3(2x+1)+5$	Substitute each x with y and y with x.
$x=\log_3(2y+1)+5$	Isolate the logarithmic term.
$x-5=\log_3(2y+1)$	
$3^{x-5}=3^{\log_3(2y+1)}$	Use the property $b^{\log_b x}=x$ to eliminate the logarithm.
$3^{x-5}=2y+1$	
$3^{x-5}-1=2y$	Solve for y.
$\dfrac{3^{x-5}-1}{2}=y$	
$\dfrac{3^{x-5}-1}{2}=f^{-1}(x)$	

c.	
$y=5^{x-6}+1$	Substitute each x with y and y with x.
$x=5^{y-6}+1$	Isolate the exponential term.
$x-1=5^{y-6}$	
$\log_5(x-1)=\log_5(5^{y-6})$	Use the property $\log_b b^x=x$ to eliminate the base of the exponent.
$\log_5(x-1)=y-6$	
$\log_5(x-1)+6=y$	Solve for y.
$\log_5(x-1)+6=f^{-1}(x)$	

d.	
$y=5\cdot 2^{3-x}-4$	Substitute each x with y and y with x.
$x=5\cdot 2^{3-y}-4$	Isolate the exponential term.
$x+4=5\cdot 2^{3-y}$	Use the property $\log_b b^x = x$ to eliminate the base of the exponent.
$\frac{x+4}{5}=2^{3-y}$	
$\log_2 \frac{x+4}{5}=\log_2 2^{3-y}$	Solve for y.
$\log_2 \frac{x+4}{5}=3-y$	
$\log_2 \frac{x+4}{5}-3=-y$	
$-\log_2 \frac{x+4}{5}+3=y$	
$3-\log_2 \frac{x+4}{5}=f^{-1}(x)$	

Exercises G

Find the inverse of each function.

1. $f(x)=\log(x+7)-2$
2. $f(x)=\log_6(x-10)-3$
3. $f(x)=2\ln(8-x)+5$
4. $f(x)=\log_3(3x-4)+1$
5. $f(x)=\log_2\left(\frac{1}{3}x+2\right)+7$
6. $f(x)=\log_4(1-2x)-3$
7. $f(x)=5^{x-3}+2$
8. $f(x)=\frac{1}{2}e^{3-x}+6$
9. $f(x)=7^{x-2}-3$
10. $f(x)=e^{4x-5}-7$
11. $f(x)=-2\cdot 3^{5-2x}+1$
12. $f(x)=\frac{1}{3}\cdot 2^{3x+4}-5$

Using the Structure of Expressions to Solve Equations (Honors)

Example 12:
Solve the equation $e^{4x} - 3e^{2x} - 18 = 0$.

$e^{4x} - 3e^{2x} - 18 = 0$	
$u^2 - 3u - 18 = 0$ $(u-6)(u+3) = 0$ $u - 6 = 0 \quad u + 3 = 0$ $u = 6 \quad u = -3$	The equation is quadratic in nature, let $u = e^{2x}$. Rewrite the equation in terms of *u*. Solve for *u*.
$e^{2x} = 6 \quad e^{2x} = -3$ $\ln e^{2x} = \ln 6$ $2x = \ln 6$ $x = \frac{\ln 6}{2}$	Substitute $e^{2x} = u$ and solve for *x*. An exponential function will never equal a negative number.

Practice Exercises H

Solve each equation.

1. $e^{2x} - 2e^x - 3 = 0$
2. $e^{4x} + 5e^{2x} - 24 = 0$
3. $3^{2x} + 3^x - 2 = 0$
4. $2^{2x} + 2^x - 12 = 0$
5. $e^{2x} - 3e^x + 2 = 0$
6. $4^{2x} + 4^x - 20 = 0$
7. $7^{2x} - 7^x - 30 = 0$
8. $e^{2x} - 10e^x + 21 = 0$
9. $5^{2x} + 5^x - 6 = 0$

Trigonometry

Unit 3 Cluster 1 (G.SRT.9): Area of a Triangle

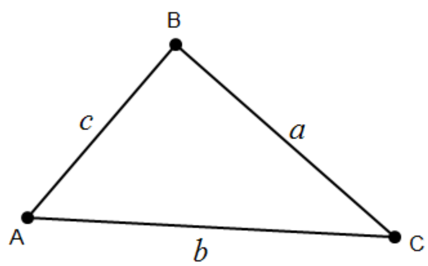
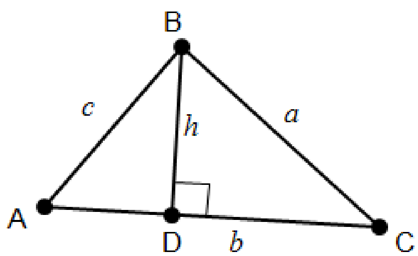
Cluster 1: Apply trigonometry to general triangles

3.1 Derive the formula for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

3.1 Find the area of triangles using the formula.

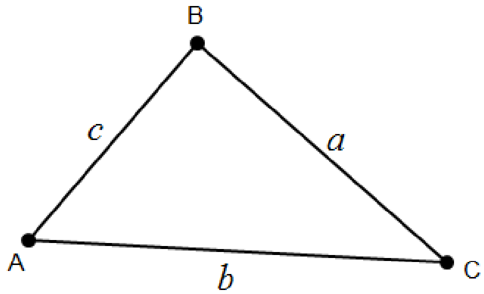
It is possible to find the area of a triangle using trigonometry when given two sides and the included angle. In order to do this, you must draw an altitude from the vertex of the non-included angle to the side opposite the angle.

Derivation of the area of a triangle

	Start with any triangle that has angles A, B, and C and side lengths, *a*, *b*, and *c*, where *a* is the side opposite angle A, *b* is the side opposite angle B, and *c* is the side opposite angle C.
	Construct an altitude, *h*, from the vertex of one of the angles to the side opposite the angle. The two triangles formed, $\triangle ABD$ and $\triangle BDC$, are right triangles.
$\sin A = \frac{h}{c}$ $c\sin A = h$	Find the measure of *h* in terms of $\angle A$ and side *c* using the sine ratio for $\angle A$.
$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$ $\text{Area} = \frac{1}{2}b(c\sin A)$ $\text{Area} = \frac{1}{2}bc(\sin A)$	The base of the entire triangle is *b* and the height is *h*. Substitute $c\sin A = h$ into the area formula for *h*, the height. NOTE: This formula works for acute, obtuse, and right triangles.

Area of a Triangle Given Two Sides and the Included Angle

The area of a triangle is half the product of the lengths of two sides and the sine of the included angle.



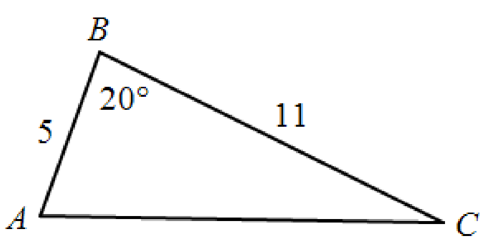
$$\text{Area} = \frac{1}{2}bc(\sin A)$$

$$\text{Area} = \frac{1}{2}ac(\sin B)$$

$$\text{Area} = \frac{1}{2}ab(\sin C)$$

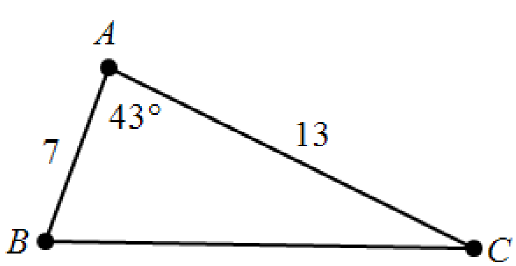
Example 1:

Find the area of a triangle with sides a = 11, b = 5, and $m\angle C = 20°$. Round your answer to the nearest thousandth (3 decimal places).

	Draw the triangle.
$\text{Area} = \frac{1}{2}ab(\sin C)$ $\text{Area} = \frac{1}{2}(11)(5)(\sin 20°)$ $\text{Area} = \frac{55}{2}(\sin 20°) \approx 9.406$	Use the formula for the area of a triangle. Substitute in known values and simplify.

Example 2:

Find the area of a triangle with sides b = 13, c = 7, and $m\angle A = 43°$. Round your answer to the nearest thousandth (3 decimal places).

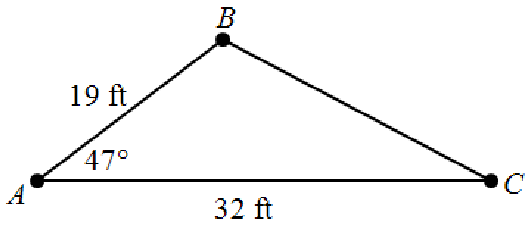
	Draw the triangle.

$\text{Area} = \frac{1}{2}bc(\sin A)$ $\text{Area} = \frac{1}{2}(13)(7)(\sin 43°)$ $\text{Area} = \frac{91}{2}(\sin 43°) \approx 31.031$	Use the formula for the area of a triangle. Substitute in known values and simplify.

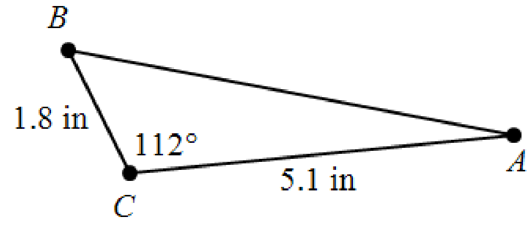
Practice Exercises A

Find the area of each triangle. Round your answer to the nearest thousandth (3 decimal places).

1.



2.



3. $B = 36°$, $a = 3$ cm, $c = 6$ cm

4. $A = 48°$, $b = 20$ m, $c = 40$ m

5. $B = 33°$, $a = 12$ ft, $c = 5.5$ ft

6. $C = 102°$, $a = 16$ mm, $b = 20$ mm

7. $A = 76°$, $b = 11$ m, $c = 24$ m

8. $B = 101°$, $a = 10$ cm, $c = 22$ cm

9. A surveyor wants to mark off a triangular parcel with an area of 0.5 acres (1 acre is equivalent to 43,560 ft^2). One side of the triangle extends 220 feet along a straight road. A second side extends at an angle of $75°$ from one end of the first side. How long should the second side be?

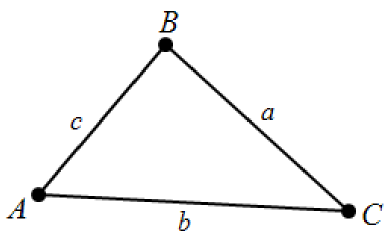
Unit 3 Cluster 1 (G.SRT.10 and G.SRT.11): The Law of Sines and Law of Cosines

Cluster 1: Apply trigonometry to general triangles

3.1 Prove the Law of Sines.
3.1 Prove the Law of Cosines.
3.1 Use the Law of Sines and the Law of Cosines to solve problems.

Law of Sines

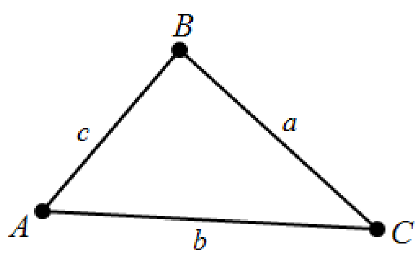
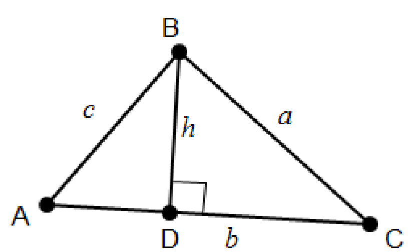
For any $\triangle ABC$, the Law of Sines relates the sine of each angle to the length of the side opposite the angle.



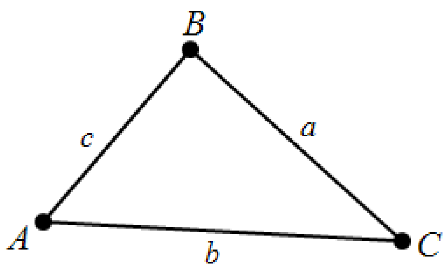
$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

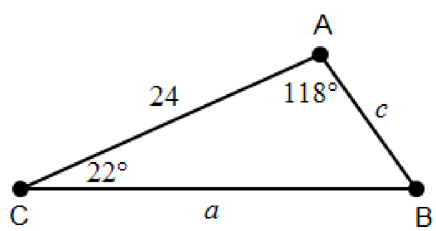
Proof:

	Start with any triangle that has angles A, B, and C and side lengths, *a*, *b*, and *c*, where *a* is the side opposite angle A, *b* is the side opposite angle B, and *c* is the side opposite angle C.
	Construct an altitude, *h*, from one of the angles to the side opposite the angle. The two triangles formed, $\triangle ABD$ and $\triangle BDC$, are right triangles.
$\sin A=\frac{h}{c}$ and $\sin C=\frac{h}{a}$	Use the definition of sine to relate the base angles, $\angle A$ and $\angle C$, to the hypotenuse of each and the altitude.
$c\sin A=h$ and $a\sin C=h$	Use the multiplication property of equality to solve each equation for *h*.
$c\sin A=a\sin C$	Use the transitive property of equality to set the equations equal to one another.
$\frac{\sin A}{a}=\frac{\sin C}{c}$	Use the division property of equality.

You can also derive the Law of Sines from the formula for the area of a triangle given two sides and the included angle.

	All three of these formulas will give the same area for the triangle. $\text{area} = \frac{1}{2}bc(\sin A)$ $\text{area} = \frac{1}{2}ac(\sin B)$ $\text{area} = \frac{1}{2}ab(\sin C)$
$\frac{1}{2}bc(\sin A) = \frac{1}{2}ac(\sin B)$	Set the right side of the first two equations equal to one another.
$bc(\sin A) = ac(\sin B)$ $b(\sin A) = a(\sin B)$	Multiply each side of the equation by 2. Divide each side of the equation by *c*.
$\frac{\sin A}{a} = \frac{\sin B}{b}$	Divide each side by *ab*.

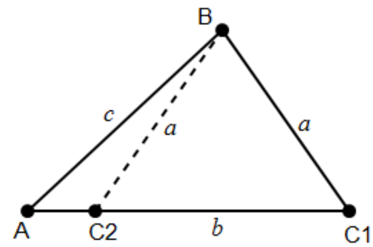
Example 1:
Use the Law of Sines to solve the triangle. Round your answers to three decimal places.



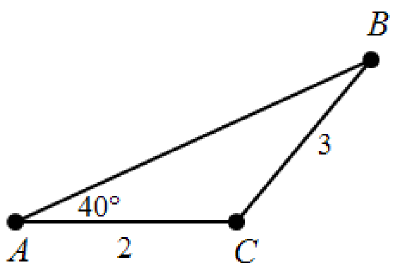
$\frac{a}{\sin A} = \frac{b}{\sin B}$ $\frac{a}{\sin 118°} = \frac{24}{\sin 40°}$	You are given two angles and a side. Use $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ to find the missing side *a*. Substitute in known values. (Hint: $m\angle B = 180° - 118° - 22° = 40°$)
$a = \frac{24}{\sin 40°}(\sin 118°)$ $a \approx 32.967$	Solve for *a*.
$\frac{c}{\sin C} = \frac{b}{\sin B}$ $\frac{c}{\sin 22°} = \frac{24}{\sin 40°}$	Use $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ again to find the missing side *c*.
$c = \frac{24}{\sin 40°}(\sin 22°)$ $c \approx 13.987$	Solve for *c*.
$a \approx 32.967$, $c \approx 13.987$, and $B = 40°$	

The Ambiguous Case (SSA)

If you are given two angles and one side (ASA or AAS), the Law of Sines will easily provide ONE solution for a missing side. However, the Law of Sines has a problem dealing with SSA. If you are given two sides and one angle, where you must find an angle, the Law of Sines could possibly provide you with one or more solutions or even no solution at all.



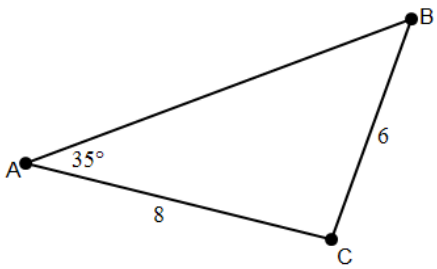
Example 2:
Use the Law of Sines to solve the triangle.

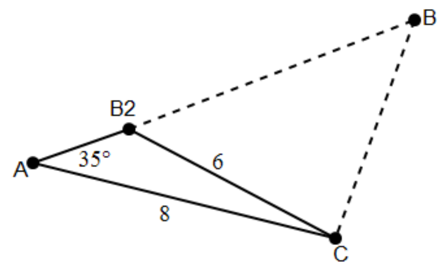


$\frac{\sin 40°}{3}=\frac{\sin B}{2}$ $\frac{2\sin 40°}{3}=\sin B$ $\sin^{-1}\left(\frac{2\sin 40°}{3}\right)=m\angle B$ $m\angle B \approx 25.374°$	Use the Law of Sines to find $m\angle B$.
$180°$ $-\ \ 25.374°$ $154.626°$ $+\ \ 40°$ $194.626°$ $194.626° > 180°$ There is only one triangle for the given information	Now check to see if more than one triangle exists with the given information. **TEST** 180 ° - (the found angle) Answer + (given angle) If sum > 180 ° there is only one triangle If sum < 180 ° there are two triangles
$m\angle C \approx 180° - (40° + 25.375°)$ $m\angle C \approx 114.625°$	Find $m\angle C$.
$\frac{3}{\sin 40°}=\frac{c}{\sin 114.625°}$	Use the Law of Sines to find side c.

$c = \dfrac{2\sin 114.625^\circ}{\sin 40^\circ}$ $c \approx 4.243$	
There is only 1 triangle: $c \approx 4.243,\ B \approx 25.374^\circ,$ and $C \approx 114.625^\circ$	

Example 3:
Use the Law of Sines to solve the triangle.
Triangle ABC with sides a = 6, b = 8, and $m\angle A = 35^\circ$.

	Draw the triangle, notice that there are two given sides and one angle.
$\dfrac{\sin 35^\circ}{6} = \dfrac{\sin B}{8}$ $\dfrac{8\sin 35^\circ}{6} = \sin B$ $\sin^{-1}\left(\dfrac{8\sin 35^\circ}{6}\right) = m\angle B$ $m\angle B \approx 49.886^\circ$	Use the Law of Sines to find $m\angle B$.
180° $-\ 49.886^\circ$ 130.114° $+\ 35^\circ$ 165.114° $165.114^\circ < 180^\circ$ There are two triangles for the given information	Now check to see if more than one triangle exists with the given information. **TEST** 180 ° - (the found angle) Answer + (given angle) If sum > 180 ° there is only one triangle If sum < 180 ° there are two triangles
$m\angle C \approx 180^\circ - (35^\circ + 49.886^\circ)$ $m\angle C \approx 95.114^\circ$	Find $m\angle C$.
$\dfrac{6}{\sin 35^\circ} = \dfrac{c}{\sin 95.114^\circ}$ $c = \dfrac{6\sin 95.114^\circ}{\sin 35^\circ}$ $c \approx 10.419$	Use the Law of Sines to find side c.

The information given can form two different triangles. We draw the second triangle by swinging $\overline{BC}$ towards $\angle A$ forming an isosceles triangle and creating $\triangle AB_2C$.	
$m\angle AB_2C = 180° - m\angle BB_2C$ $m\angle AB_2C = 180° - 49.886°$ $m\angle AB_2C = 130.114°$	Since $\triangle BB_2C$ is an isosceles triangle, $\angle BB_2C \cong \angle B$. $\angle AB_2C$ forms a straight angle with $\angle BB_2C$.
$m\angle ACB_2 = 180° - (35° + 130.114°)$ $m\angle ACB_2 = 14.886°$	Find $\angle ACB_2$.
$\dfrac{6}{\sin 35°} = \dfrac{\overline{AB_2}}{\sin 14.886°}$ $\dfrac{6\sin 14.886°}{\sin 35°} = \overline{AB_2}$ $\overline{AB_2} \approx 2.687$	Use the Law of Sines to find $\overline{AB_2}$.
Triangle 1: $c \approx 10.419$, $B \approx 49.886°$, and $C \approx 95.114°$ Triangle 2: $c \approx 2.687$, $B \approx 130.114°$, and $C \approx 14.886°$	

Practice Exercises A

Use the Law of Sines to solve each triangle. Round each answer to the nearest thousandth.

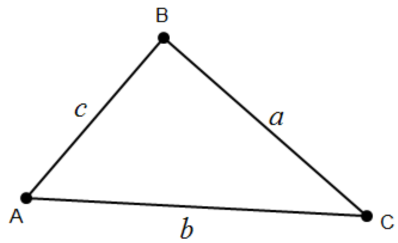
1. $A = 40°,\ B = 30°,\ b = 10$
2. $A = 70°,\ C = 62°,\ a = 7.3$
3. $A = 60°,\ B = 45°,\ b = 3.7$
4. $B = 16°,\ C = 103°,\ c = 12$
5. $B = 62°,\ C = 41°,\ a = 14$
6. $A = 100°,\ C = 35°,\ a = 22$
7. $A = 40°,\ a = 20,\ b = 15$
8. $B = 70°,\ b = 14,\ c = 9$
9. $C = 50°,\ b = 20,\ c = 30$
10. $C = 112°,\ a = 37,\ c = 42.1$
11. $A = 49°,\ a = 32,\ b = 28$
12. $B = 103°,\ b = 61,\ c = 46$

If possible, use the Law of Sines to solve each triangle. There may be one triangle, two triangles, or no triangles at all. Round each answer to the nearest thousandth.

13. $A = 80°,\ a = 17,\ c = 14$
14. $A = 150°,\ a = 9.3,\ b = 41$
15. $C = 36°,\ a = 17,\ c = 16$
16. $B = 60°,\ a = 18,\ b = 16$
17. $C = 30°,\ b = 40,\ c = 10$
18. $B = 30°,\ a = 18,\ b = 9$
19. $C = 38°,\ b = 25,\ c = 21$
20. $B = 38°,\ a = 18,\ b = 12$
21. $A = 63°,\ a = 10,\ c = 8.9$

Law of Cosines

For any $\triangle ABC$, the Law of Cosines relates the length of a side to the other two sides of a triangle and the cosine of the included angle.

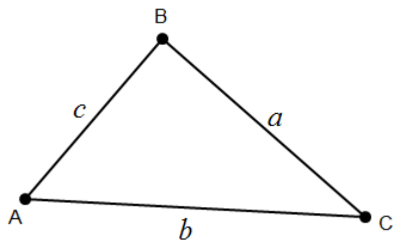
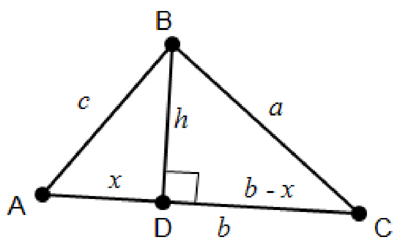


$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

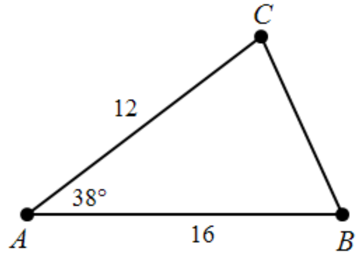
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof:

	Start with any triangle that has angles A, B, and C and side lengths, *a*, *b*, and *c*, where *a* is the side opposite angle A, *b* is the side opposite angle B, and *c* is the side opposite angle C.
	Construct an altitude, *h*, from one of the angles to the side opposite the angle. The two triangles formed, $\triangle ABD$ and $\triangle BDC$, are right triangles.
$a^2 = (b-x)^2 + h^2$	Use the Pythagorean theorem to relate the sides of $\triangle BDC$.
$a^2 = b^2 - 2bx + x^2 + h^2$	Expand the binomial.
$a^2 = b^2 - 2b\left[c(\cos A)\right] + c^2$	Use $\cos A = \frac{x}{c}$ or $c(\cos A) = x$ and $x^2 + h^2 = c^2$.
$a^2 = b^2 + c^2 - 2bc(\cos A)$	Rearrange the terms.

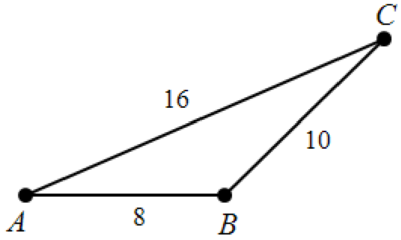
Example 4:

Solve the triangle. Round your answers to the nearest thousandth.



$a^2 = 12^2 + 16^2 - 2(12)(16)(\cos 38°)$	Use $a^2 = b^2 + c^2 - 2bc\cos A$. Substitute in known values.
$a^2 = 144 + 256 - 384(\cos 38°)$ $a^2 = 400 - 384(\cos 38°)$ $a^2 \approx 97.404$ $a \approx 9.8693$	Simplify.
$\frac{\sin A}{a} = \frac{\sin B}{b}$ $\frac{\sin 38°}{9.8693} = \frac{\sin B}{12}$ $\frac{12\sin 38°}{9.8693} = \sin B$ $\sin^{-1}\left(\frac{12\sin 38°}{9.8693}\right) = m\angle B$ $48.467° \approx m\angle B$	Use the Law of Sines to find one of the missing angles. Substitute in known values. Simplify.
$m\angle C \approx 180° - 38° - 48.467°$ $m\angle C \approx 93.533°$	Find the third angle.
$a \approx 9.870,\ B \approx 48.467°$, and $C \approx 93.533°$	

Example 5:
Solve the triangle. Round your answers to the nearest thousandth.



$b^2 = a^2 + c^2 - 2ac\cos B$ $16^2 = 10^2 + 8^2 - 2(10)(8)(\cos B)$	Use the Law of Cosines to find the angle opposite the **longest** side. Angle B is opposite the **longest** side. Substitute in known values.
$256 = 100 + 64 - 160(\cos B)$ $256 = 164 - 160(\cos B)$ $92 = -160(\cos B)$ $-0.575 = \cos B$ $\cos^{-1}(-0.575) = m\angle B$ $125.100 \approx m\angle B$	Simplify.

$$\frac{\sin 125.1°}{16} = \frac{\sin A}{10}$$ $$\frac{10\sin 125.1°}{16} = \sin A$$ $$\sin^{-1}\left(\frac{10\sin 125.1°}{16}\right) = m\angle A$$ $$30.753° \approx m\angle A$$	Use the Law of Sines to solve for either of the two remaining angles.
$m\angle C \approx 180° - 125.1° - 30.753°$ $m\angle C \approx 24.147°$	Find the third angle.
$A \approx 30.753°$, $B \approx 125.100°$, and $C \approx 24.147°$	

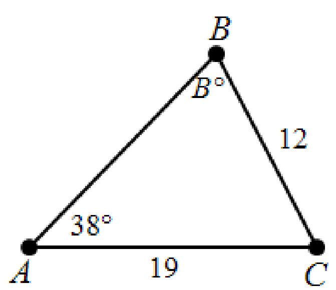
Practice Exercises B

Solve the triangle. Round your answer to the nearest thousandth.

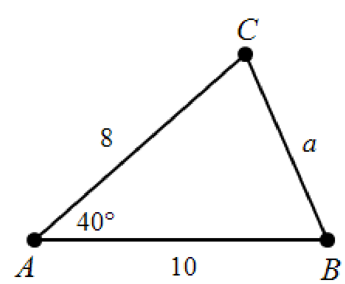
1. $C = 27°$, $a = 5$, $b = 9$
2. $A = 100°$, $b = 4$, $c = 1$
3. $B = 40°$, $a = 80$, $c = 78$
4. $B = 35°$, $a = 43$, $c = 19$
5. $C = 42°$, $a = 5$, $b = 7$
6. $A = 55°$, $b = 12$, $c = 7$
7. $a = 2$, $b = 5$, $c = 4$
8. $a = 10$, $b = 12$, $c = 21$
9. $a = 5$, $b = 7$, $c = 10$
10. $a = 3$, $b = 3$, $c = 5$
11. $a = 11$, $b = 14$, $c = 19$
12. $a = 4$, $b = 7$, $c = 6$

Determine if you would use the Law of Sines or the Law of Cosines to find missing value. Find the missing value. Round your answer to the nearest thousandth.

13.

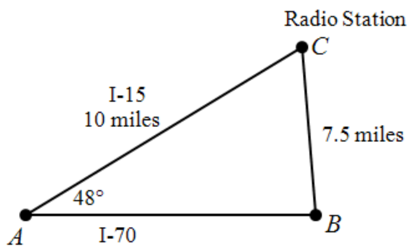
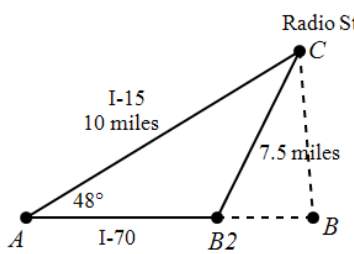


14.



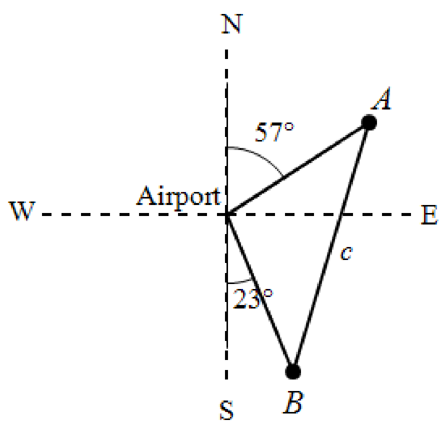
15. $b = 4$, $c = 8$, $A = 46°$; find a.
16. $a = 10$, $c = 8.9$, $A = 63°$; find b.
17. $a = 4$, $A = 80°$, $C = 15°$; find c.
18. $a = 12$, $b = 21$, $C = 95°$; find c.
19. $a = 18$ $b = 17$, $c = 12$; find $m\angle C$.
20. $a = 14$, $B = 41°$, $C = 62°$; find b.

Example 6:
A radio station located adjacent to I-15 is 10 miles from where it connects to I-70. The angle between the two interstates is $48°$. The station can broadcast for a range of 7.5 miles. If Bryce is driving on I-70, between what two distances from the intersection of the two highways can he receive the radio signal?

	Draw a picture to represent the situation. You need to find the length of $\overline{AB}$ in order to find when Bryce can hear the radio station.
$\frac{\sin 48°}{7.5}=\frac{\sin B}{10}$ $\frac{10\sin 48°}{7.5}=\sin B$ $\sin^{-1}\left(\frac{10\sin 48°}{7.5}\right)=m\angle B$ $82.2474°\approx m\angle B$	Use the Law of Sines to solve for angle B.
$m\angle C=180°-(48°+82.2474°)=49.7526°$	Find the measure of angle C.
$\frac{7.5}{\sin 48°}=\frac{AB}{\sin 49.7526°}$ $\frac{7.5\sin 49.7526°}{\sin 48°}=AB$ $7.703\approx AB$	Use the Law of Sines to find the length of $\overline{AB}$.
$180°$ $-\ 82.2474°$ $97.7526°$ $+\ 48°$ $145.7526°$ $145.7526°<180°$ There are two triangles for the given information	Now check to see if more than one triangle exists with the given information. **TEST** 180 ° - (the found angle) Answer + (given angle) If sum > 180 ° there is only one triangle If sum < 180 ° there are two triangles
	Two triangles can be formed. Bryce will be able to hear the radio station when he is between *B*2 and *B*.

$m\angle AB_2C = 180° - m\angle BB_2C$ $m\angle AB_2C = 180° - 82.2474°$ $m\angle AB_2C = 97.7526°$	Since $\triangle BB_2C$ is an isosceles triangle, $\angle BB_2C \cong \angle B$. $\angle AB_2C$ forms a straight angle with $\angle BB_2C$.
$m\angle C = 180° - (48° + 97.7526°) = 34.2474°$	Find the measure of angle C in the new triangle.
$\frac{7.5}{\sin 48°} = \frac{AB_2}{\sin 34.2474°}$ $\frac{7.5 \sin 34.2474°}{\sin 48°} = AB_2$ $5.67959 \approx AB_2$	Use the Law of Sines to find the length of $\overline{AB_2}$.
Bryce can hear the radio station when he is 5.7 to 7.7 miles from where the two interstates intersect on I-70.	

Example 7:
Two airplanes leave an airport at the same time on different runways. One flies on a bearing of $N57°E$ ($57°$ east of north) at 320 miles per hour. The other airplane flies on a bearing of $S23°E$ ($23°$ east of south) at 310 miles per hour. How far apart will the airplanes be after 1.5 hours?

	Draw a triangle to represent the situation. The angle between the two airplanes is $m\angle C = 180° - (57° + 23°) = 100°$. The distance the first plane has traveled after 1.5 hours is $b = 320(1.5) = 480$ miles. The distance the second plane has traveled after 1.5 hours is $a = 310(1.5) = 465$ miles.
$c^2 = 465^2 + 480^2 - 2(465)(480)\cos 100°$ $c^2 = 216,225 + 230,400 - 446,400 \cos 100°$ $c^2 = 446,625 - 446,400 \cos 100°$ $c = \sqrt{446,625 - 446,400 \cos 100°}$ $c \approx 723.976$	Use the Law of Cosines to find the measure of side *c*, which is the distance between the two planes.
The planes are about 724 miles apart after 1.5 hours.	

Practice Exercises C

1. Lighthouse B is 8 miles east of lighthouse A. A boat leaves A and sails 6 miles. At this time, it is sighted from B. If the bearing of the boat from lighthouse B is $S71°W$, how far from lighthouse B is the boat? Round your answer to the nearest mile.

2. You and a friend hike 1.3 kilometers due west from a campsite. At the same time, two other friends hike 1.7 kilometers at a heading of $N17°W$ from the campsite. To the nearest tenth of a kilometer, how far apart are the two groups?

3. An air traffic controller is tracking a plane 2.3 miles due north of the radar tower. A second plane is located 3.6 miles from the tower at a heading of $S72°W$. To the nearest tenth of a mile, how far apart are the two planes?

4. Two observers are 450 feet apart on opposite sides of a flagpole. The angles of elevation from the observers to the top of the pole are $23°$ and $25°$. Find the height of the flagpole to the nearest foot.

5. Two fire-lookout stations are 15 miles apart, with station B directly west of station A. Both stations spot a fire. The bearing of the fire from station A is $S28°W$ and the bearing of the fire from station B is $S49°E$. How far, to the nearest tenth of a mile, is the fire from each lookout station?

6. The player waiting to receive a kickoff stands at the 7 yard line (point A) as the ball is being kicked 61 yards up the field from the opponent's 32 yard line. The kicked ball travels 69 yards at an angle of 10 degrees to the right of the receiver (point B). Find the distance the receiver runs to catch the ball.

7. The dimensions of a triangular flag are 15 inches by 24 inches by 29 inches. To the nearest tenth, what is the measure of the angle formed by the two shorter sides?

8. A leaning wall is inclined at $4°$ from vertical. At a distance of 25 feet from the wall, the angle of elevation to the top is $34°$. Find the height of the wall to the nearest tenth of a foot.

9. A 25-ft water slide has a 10.8-ft. ladder which meets the slide at a $100°$ angle. To the nearest tenth, what is the distance between the end of the slide and the bottom of the ladder?

10. Two observers are 2.4 miles apart on opposite sides of a hot-air balloon. The angle of elevation from observer A is $30°$ and the angle of elevation from observer B is $35°$. Find the altitude of the balloon to the nearest tenth of a mile.

11. After a wind storm, you notice that your 12-foot flagpole may be leaning, but you are not sure. From a point on the ground 10 feet from the base of the flagpole, you find that the angle of elevation to the top is $52°$. Find the angle, to the nearest degree, that the flagpole makes with the ground and determine if it is leaning or not.

12. Two airplanes flying together in formation take off in different directions. One flies due east at 340 mph, and the other flies $N12°E$ at 360 mph. To the nearest tenth, how far apart are the two airplanes 1 hour after they separate, assuming that they fly at the same altitude?

13. One side of a ravine is 18 feet long. The other side is 13 feet long. A 24 foot zipline runs from the top of one side of the ravine to the other. To the nearest tenth, at what angle do the sides of the ravine meet?

14. Two ships leave a harbor at the same time. One ship travels on a bearing of $N\,14°\,E$ at 12 miles per hour. The other ship travels on a bearing of $S\,74°\,W$ at 9 miles per hour. To the nearest tenth of a mile, how far apart will the ships be after three hours?

15. A surveyor needs to determine the distance between two points that lie on opposite banks of a river. Two points, A and C, along one bank are 250 yards apart. The point B is on the opposite bank. Angle A is $64°$ and angle C is $51°$. Find the distance between A and B to the nearest tenth of a yard.

16. The FCC is attempting to locate an illegal radio station. It sets up two monitoring stations, A and B, with station B 30 miles east of station A. Station A measures the illegal signal from the radio station as coming from a direction of $42°$ east of north. Station B measures the signal as coming from a point $40°$ west of north. How far is the illegal radio station from monitoring stations A and B?

Unit 3 Cluster 2 (F.TF.1, F.TF.2, and F.TF.3): The Unit Circle

Cluster 2: Extending the domain of trigonometric functions using the unit circle

3.2 Understand radian measure as the length of the arc on the unit circle subtended by the angle.

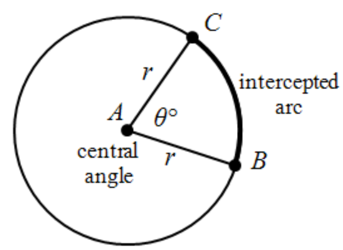
3.2 Use special triangles to determine geometrically the values of sine, cosine, and tangent for $\frac{\pi}{3}, \frac{\pi}{4}$, and $\frac{\pi}{6}$.

3.2 Use the unit circle to express values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$.

VOCABULARY

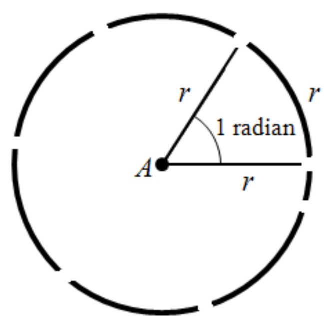
An angle with its vertex at the center of the circle is called a **central angle**.

An **intercepted arc** is the portion of a circle with endpoints on the sides of the central angle and remaining points within the interior of the angle.



A **radian** is the measure of the central angle that intercepts an arc with length equal to the radius of the circle. You can see that it takes 6 radians and a little more (about 0.28) to complete the entire circle. Mathematically, $\frac{C}{r} = \frac{2\pi r}{r} = 2\pi$. Therefore, there are about 6.28 radii around a circle or exactly 2π radians.

A radian, much like an angle in degrees, measures the amount of rotation from the initial side to the terminal side of an angle in terms of the radius.



Recall that the length of the arc intercepted by an angle is proportional to the radius. The formula for the length of an intercepted arc is $l = \frac{\pi\theta}{180°}r$, where θ is measured in degrees and $\frac{\pi}{180°}$ is the constant of proportionality. The radian measure of the angle of the intercepted arc was defined to be the constant of proportionality. Using this fact we can find the measure of the angle whose intercepted arc is *r* and prove that it is 1 radian.

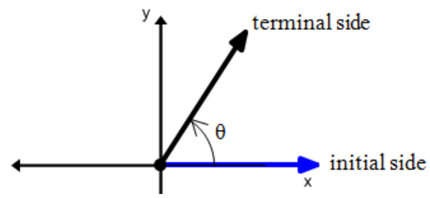
$l = \frac{\pi\theta}{180°}r$	The formula for the length of an intercepted arc.
$r = \frac{\pi\theta}{180°}r$	The length of the intercepted arc is equal to *r*.
$1 = \frac{\pi\theta}{180°}$	Divide each side by *r*.

$1 = \frac{\pi}{180°}\theta$ $1 = \theta$ radian	The constant of proportionality $\frac{\pi}{180°}$ is defined to be a radian measure.
The measure of the angle whose intercepted arc is equal to the length of the radius of the circle is 1 radian.	

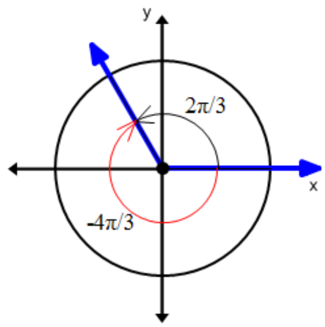
Because all circles are similar, an angle measuring 1 radian will always subtend (intercept) an arc that is equal to 1 radius of the circle.

VOCABULARY

An angle is in **standard position** when the vertex is at the origin and the initial side is on the positive x-axis.



Coterminal angles are angles with the same initial and terminal sides, but different measures.

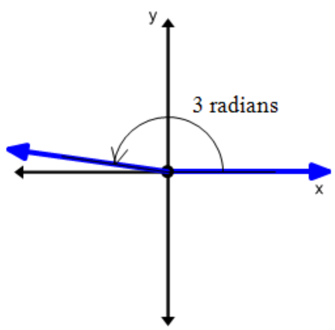
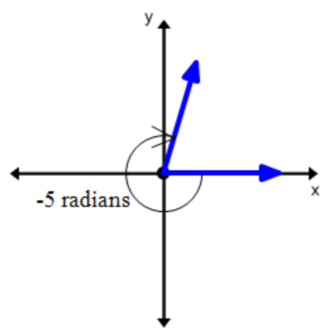


For example $\frac{\pi}{6}$ and $-\frac{11\pi}{6}$ are coterminal angles, as well as $\frac{\pi}{2}$ and $\frac{5\pi}{2}$.

Example 1: Sketch each angle in standard position.

a. 3 radians

b. -5 radians

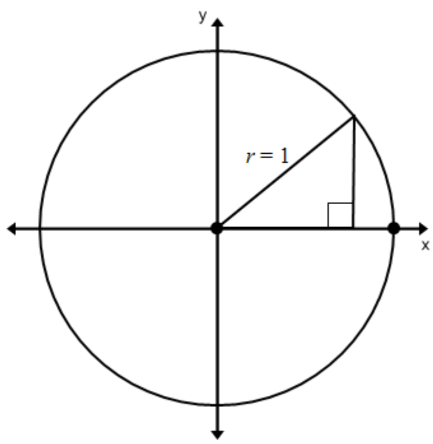
a.	b.
	

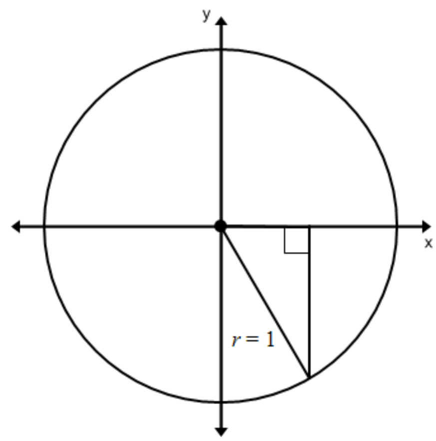
Practice Exercises A

Sketch each angle in standard position.

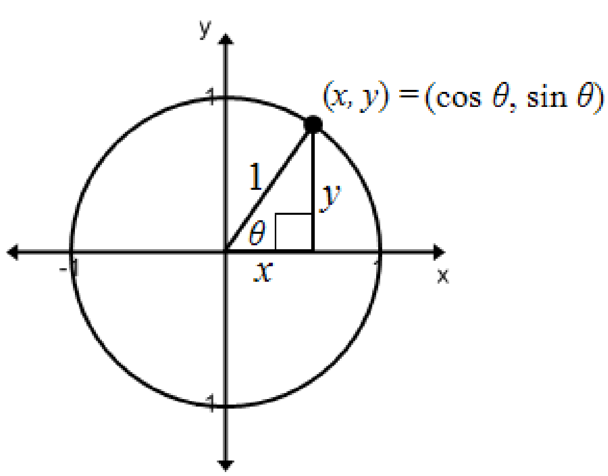
1. 2 radians
2. 7 radians
3. 4.5 radians
4. -4 radians
5. -10 radians
6. -5.5 radians

A circle with radius 1 is called a unit circle. The unit circle provides a connection between trigonometric ratios and the trigonometric functions. We can place it on a coordinate plane and use right triangle trigonometry to find the basic trigonometric ratios. Any right triangle with hypotenuse of length 1 can be drawn in any quadrant of the unit circle.





Trigonometric Ratios for a Circle of Radius 1



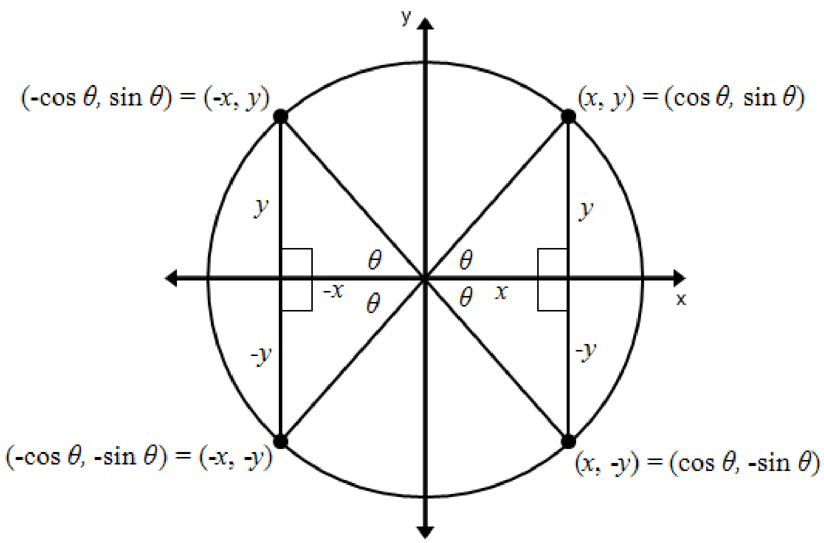
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sin\theta = \frac{y}{1} \qquad \cos\theta = \frac{x}{1}$$

$$\sin\theta = y \qquad \cos\theta = x$$

For any point (x, y) on a unit circle, the *x*-coordinate is the cosine of the angle and the *y*-coordinate is the sine of the angle. Recall that if you reflect any point (x, y) on the coordinate plane across the *x* axis, *y* axis or through the origin, then the following relationships exist:

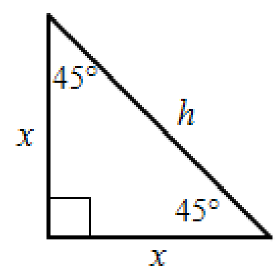
Reflection across the *y* axis $(x, y) \to (-x, y)$
Reflection across the *x* axis $(x, y) \to (x, -y)$
Reflection through the origin $(x, y) \to (-x, -y)$



There are special right triangles with special relationships between the lengths of their sides. These relationships can be used to simplify calculations when finding missing angles and sides.

45°-45°-90° Triangle

The Pythagorean Theorem allows us to derive the relationships that exist for these triangles. Consider a right isosceles triangle with leg lengths *x* and hypotenuse *h*. Since this is a right isosceles triangle then the measures of the angles are $45° - 45° - 90°$

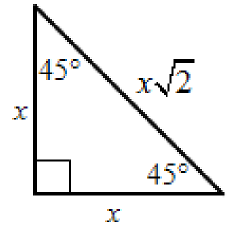


Using the Pythagorean Theorem, we know that $x^2 + x^2 = h^2$. Solving the equation for *x*, you can find the length of the legs.

$$h^2 = x^2 + x^2$$
$$h^2 = 2x^2$$
$$h = \sqrt{2x^2}$$
$$h = x\sqrt{2}$$

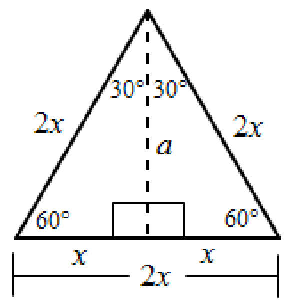
$45^\circ - 45^\circ - 90^\circ$ Triangle

In any $45^\circ - 45^\circ - 90^\circ$ triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of its leg.



30°-60°-90° Triangles

There is also a special relationship for triangles with angles of $30^\circ - 60^\circ - 90^\circ$. When an altitude, *a*, is drawn from the vertex of an equilateral triangle it bisects the base of the triangle. Two congruent $30^\circ - 60^\circ - 90^\circ$ triangles are formed. If each triangle has a base of length *x*, then the entire length of the base of the equilateral triangle is 2*x*.



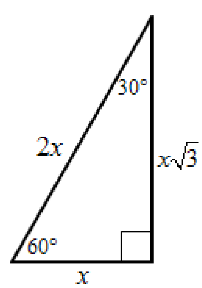
Using one of the right triangles and the Pythagorean Theorem to find the length of the altitude, *a*, we get $a^2 + x^2 = (2x)^2$. Solving it for *a* we get:

$$a^2 + x^2 = (2x)^2$$
$$a^2 = 4x^2 - x^2$$
$$a^2 = 3x^2$$
$$a = \sqrt{3x^2}$$
$$a = x\sqrt{3}$$

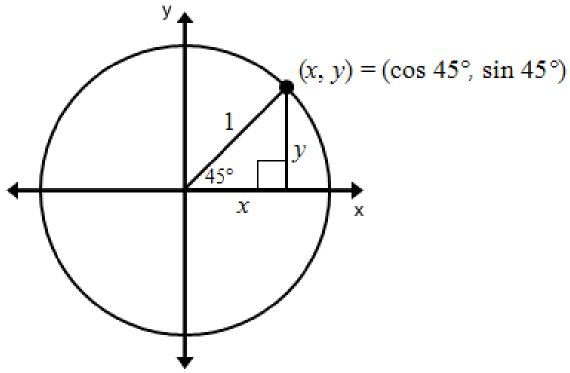
Therefore, in a $30^\circ - 60^\circ - 90^\circ$ triangle, the measures of the side lengths are x, $x\sqrt{3}$, and $2x$.

$30^\circ - 60^\circ - 90^\circ$ Triangles

In any $30^\circ - 60^\circ - 90^\circ$ triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.



The figure illustrates how these triangles can be used to derive parts of the unit circle.



Create a right triangle with central angle $45°$. The hypotenuse is length 1 and the legs are lengths *x* and *y*. The angles of the triangle measure $45°, 45°,$ and $90°$. Since two of the angles are congruent this is an isosceles triangle making the lengths of the legs the same, $x = y$. Finding the value of *x* will help us identify the numerical coordinates of the point (x, y).

By the Pythagorean Theorem we know that $x^2 + x^2 = 1^2$. Solving the equation for *x* will give us the value of each leg.

$x^2 + x^2 = 1$	Pythagorean Theorem
$2x^2 = 1$ $x^2 = \frac{1}{2}$	Isolate *x*.
$\sqrt{x^2} = \sqrt{\frac{1}{2}}$	Use properties of rational exponents to simplify.
$x = \frac{1}{\sqrt{2}}$ $x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ $x = \frac{\sqrt{2}}{2}$	Rationalize the denominator.

Since both legs are equal, $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$.

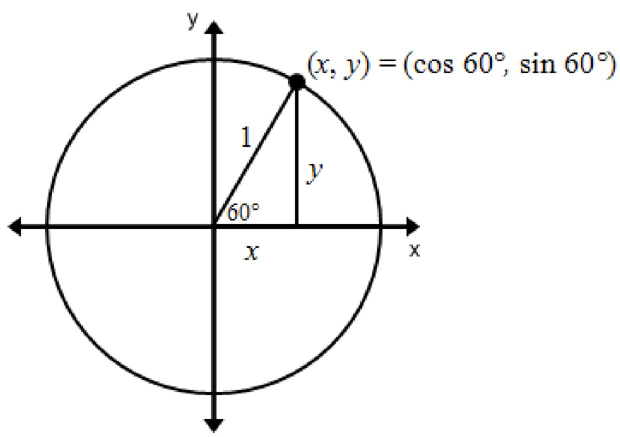
If we relate this to the unit circle where $\theta = 45°$, then the following is true:

$$y = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \sin 45° = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2} \qquad x = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \cos 45° = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

Thus, the point $(x,y)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point across the *y*-axis is equivalent to a rotation of $135°$ from the positive *x*-axis. The coordinates of the point are
  $(-x,y)=\left(-\cos 45°,\sin 45°\right)=\left(\cos 135°,\sin 135°\right)=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $225°$ from the positive *x*-axis. The coordinates of the new point are
  $(-x,-y)=\left(-\cos 45°,-\sin 45°\right)=\left(\cos 225°,\sin 225°\right)=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$.
- The reflection of the point across the *x*-axis is equivalent to a rotation of $315°$ from the positive *x*-axis. The coordinates of the new point are
  $(x,-y)=\left(\cos 45°,-\sin 45°\right)=\left(\cos 315°,\sin 315°\right)=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$.

To illustrate a $30°-60°-90°$ triangle on the unit circle, create a right triangle with central angle $60°$. The hypotenuse is length 1 and the legs are lengths *x* and *y*.



The length of the hypotenuse in a $30°-60°-90°$ is twice the length of the shorter side. Thus, $1=2x$. Upon solving the equation for *x* we find that $x=\frac{1}{2}$. We can use this value of *x* to find the length of the longer leg, *y*. The longer leg is $\sqrt{3}$ times the length of the shorter leg. Therefore, $y=x\sqrt{3}$ and since $x=\frac{1}{2}$ then $y=\left(\frac{1}{2}\right)\sqrt{3}$ or $y=\frac{\sqrt{3}}{2}$.

If we relate this to the unit circle where $\theta=°60$ then the following is true:

$$y=\sin\theta=\sin 60°=\frac{\frac{\sqrt{3}}{2}}{1}=\frac{\sqrt{3}}{2} \qquad x=\cos\theta=\cos 60°=\frac{\frac{1}{2}}{1}=\frac{1}{2}$$

Thus, the point $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point across the *y*-axis is equivalent to a rotation of $120°$ from the positive *x*-axis. The coordinates of the point are $(-x, y) = \left(-\cos 60°, \sin 60°\right) = \left(\cos 120°, \sin 120°\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $240°$ from the positive *x*-axis. The coordinates of the new point are $(-x, -y) = \left(-\cos 60°, -\sin 60°\right) = \left(\cos 240°, \sin 240°\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.
- The reflection of the point across the *x*-axis is equivalent to a rotation of $300°$ from the positive *x*-axis. The coordinates of the new point are $(x, -y) = \left(\cos 60°, -\sin 60°\right) = \left(\cos 300°, \sin 300°\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

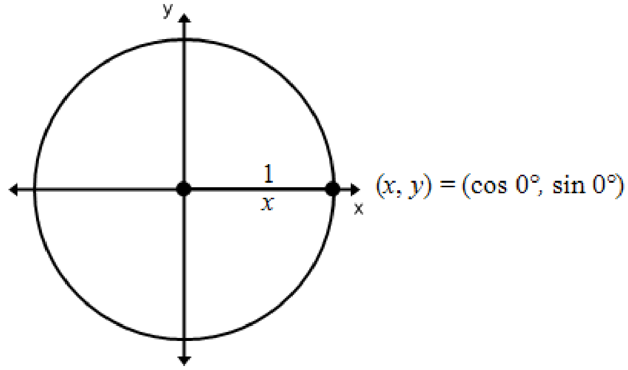
We can also consider the case where $\theta = 30°$ and use the $30° - 60° - 90°$ triangle to find the values of *x* and *y* for this value of θ.

$$y = \sin\theta = \sin 30° = \frac{\frac{1}{2}}{1} = \frac{1}{2} \qquad x = \cos\theta = \cos 30° = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

Thus, the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. If we reflect this point so that it appears in Quadrants II, III, and IV, then we can derive the following:

- The reflection of the point across the *y*-axis is equivalent to a rotation of $150°$ from the positive *x*-axis. The coordinates of the point are $(-x, y) = \left(-\cos 30°, \sin 30°\right) = \left(\cos 150°, \sin 150°\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
- The reflection of the point through the origin is equivalent to a rotation of $210°$ from the positive *x*-axis. The coordinates of the new point are $(-x, -y) = \left(-\cos 30°, -\sin 30°\right) = \left(\cos 210°, \sin 210°\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.
- The reflection of the point across the *x*-axis is equivalent to a rotation of $330°$ from the positive *x*-axis. The coordinates of the new point are $(x, -y) = \left(\cos 30°, -\sin 30°\right) = \left(\cos 330°, \sin 330°\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

If we plot all of the points where $\theta = 30°, 45°, \text{ and } 60°$ and their reflections, then we get most of the unit circle. To obtain the rest of the unit circle we have to examine what happens to a point when $\theta = 0°$.

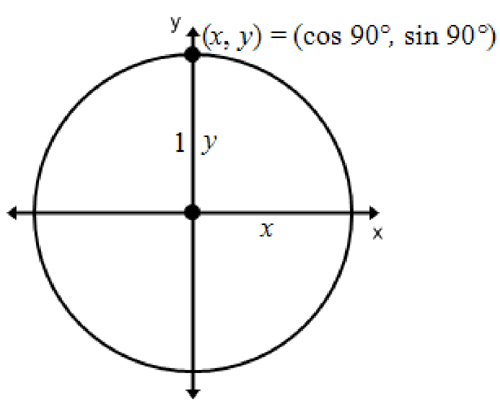


Notice that the value of *x* is equal to 1 and that *y* is zero. Thus, when $\theta = 0°$, the point $(x, y) = (1, 0)$. Even though this point does not form a right triangle, any point on a circle can be found by using cosine and sine. Therefore, $\cos 0° = 1$ and $\sin 0° = 0$.

If we reflect the point across the *y*-axis, then the new point is $(-x, y) = (-1, 0)$. This is equivalent to a rotation of $180°$ from the positive *x*-axis. The coordinates of the new point are:

$$(-x, y) = (-\cos 0°, \sin 0°) = (\cos 180°, \sin 180°) = (-1, 0).$$

Finally we need to observe what happens when we rotate a point $90°$ from the positive *x*-axis.

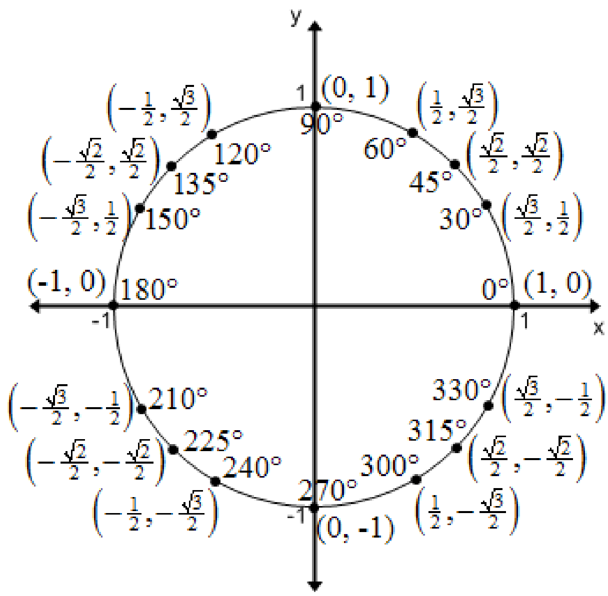


Notice that the value of *y* is equal to 1 and that *x* is zero. Thus, when $\theta = 90°$, the point $(x, y) = (0, 1)$. Therefore, $\cos 90° = 0$ and $\sin 90° = 1$.

If we reflect the point across the *x*-axis, then the new point is $(x, -y) = (0, -1)$. This is equivalent to a rotation of $270°$ from the positive *x*-axis. The coordinates of the new point are:

$$(x, -y) = (\cos 90°, -\sin 90°) = (\cos 270°, \sin 270°) = (0, -1).$$

Plotting all of the points, we obtain what is referred to as the unit circle.



The unit circle can be used to find exact values of trigonometric ratios for the angles that relate to the special right triangle angles.

Converting Between Radians and Degrees

To convert degrees to radians, multiply the angle by $\frac{\pi \text{ radians}}{180^\circ}$.

To covert radians to degrees, multiply the angle by $\frac{180^\circ}{\pi \text{ radians}}$.

The unit circle can be represented using radian measures instead of degrees.

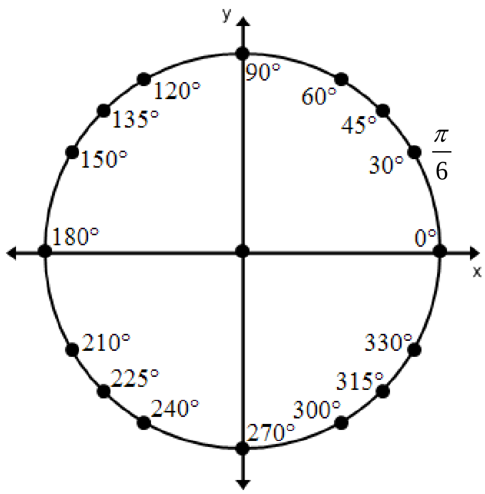
Example 2:

Convert 30° to a radian measure.

$30^\circ \cdot \frac{\pi \text{ radians}}{180^\circ}$ $\pi \text{ radians} \cdot \frac{30^\circ}{180^\circ}$	Multiply the angle by the conversion factor $\frac{\pi \text{ radians}}{180^\circ}$.
$\frac{\pi}{6}$ radians	Simplify the fraction.

Practice Exercises B

Find the radian measure for each angle listed below.



1. $0°$
2. $30° = \frac{\pi}{6}$
3. $45°$
4. $60°$
5. $90°$
6. $120°$
7. $135°$
8. $150°$
9. $180°$
10. $210°$
11. $225°$
12. $240°$
13. $270°$
14. $300°$
15. $315°$
16. $330°$

The unit circle can be used to find exact values of trigonometric ratios for the angles (degree or radian) that relate to the special right triangle angles.

Example 3:
Use the unit circle to find the exact value.

a. $\sin 135°$

b. $\cos\frac{5\pi}{4}$

Answer:

a.	The point that has been rotated $135°$ from the positive *x*-axis has coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The *y*-coordinate is the sine value, therefore, $\sin 135° = \frac{\sqrt{2}}{2}$.
b.	The point that has been rotated $\frac{5\pi}{4}$ radians from the positive *x*-axis has coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The *x*-coordinate is the cosine value, therefore, $\cos\frac{5\pi}{4} = -\frac{1}{2}$.

Defining Tangent Values

Another way to write $\tan\theta$ is $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. This can be shown algebraically as follows:

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}}$	Use the definition $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta \dfrac{\text{adjacent}}{\text{hypotenuse}}$.
$\tan\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} \div \dfrac{\text{adjacent}}{\text{hypotenuse}}$	Rewrite the division problem so that it is easier to work with.
$\tan\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} \cdot \dfrac{\text{hypotenuse}}{\text{adjacent}}$	Dividing by a fraction is the same as multiplying by its reciprocal.
$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} \cdot \dfrac{\text{hypotenuse}}{\text{hypotenuse}}$	Use the commutative property of multiplication to rearrange the terms.
$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} \cdot 1$	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{\sin\theta}{\cos\theta}$

Example 4:

Find $\tan\dfrac{7\pi}{6}$.

Answer:

The coordinates of the point that has been rotated $\dfrac{7\pi}{6}$ from the positive x-axis are $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\frac{7\pi}{6} = \frac{\sin\frac{7\pi}{6}}{\cos\frac{7\pi}{6}}$$

Use the coordinates of the point to find $\tan\dfrac{7\pi}{6}$.

$$\tan\frac{7\pi}{6} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$

$\sin\dfrac{7\pi}{6} = -\dfrac{1}{2}$ and $\sin\dfrac{7\pi}{6} = -\dfrac{\sqrt{3}}{2}$

$$\tan\frac{7\pi}{6} = -\frac{1}{2} \div -\frac{\sqrt{3}}{2}$$

Rewrite the division problem.

$$\tan\frac{7\pi}{6} = -\frac{1}{2} \cdot -\frac{2}{\sqrt{3}}$$

Dividing by a fraction is the same as multiplying by the reciprocal.

$$\tan\frac{7\pi}{6} = \frac{1}{\sqrt{3}}$$

Simplify.

$$\tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3}$$

Rationalize the denominator.

Tangent Values for the Angles on the Unit Circle

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$
θ	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$
θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$

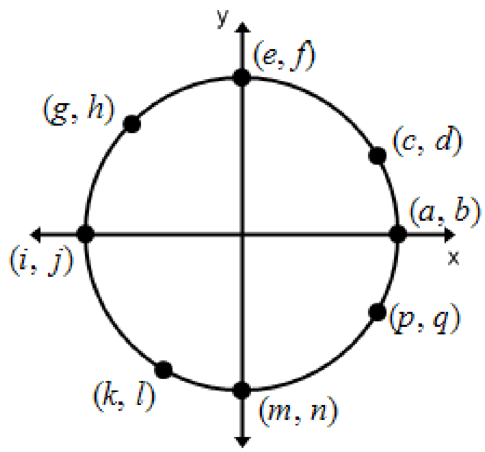
Practice Exercises C

Use the unit circle to find the exact value.

1. $\cos 135°$
2. $\tan 270°$
3. $\sin 300°$
4. $\cos 45°$
5. $\tan 60°$
6. $\sin 120°$
7. $\cos 180°$
8. $\tan 0°$
9. $\sin 210°$
10. $\cos 240°$
11. $\tan 225°$
12. $\sin 315°$
13. $\sin\frac{11\pi}{6}$
14. $\cos\frac{\pi}{6}$
15. $\tan\frac{5\pi}{6}$
16. $\sin\frac{\pi}{2}$
17. $\cos\pi$
18. $\tan\frac{7\pi}{4}$
19. $\sin\frac{5\pi}{4}$
20. $\cos\frac{7\pi}{6}$
21. $\tan\frac{3\pi}{2}$
22. $\sin\frac{\pi}{4}$
23. $\cos\frac{2\pi}{3}$
24. $\tan\frac{\pi}{2}$

Refer to the diagram. Give the letter that could stand for the function value.

25. $\sin 180°$
26. $\cos 270°$
27. $\cos 30°$
28. $\sin 135°$
29. $\cos 0°$
30. $\sin 330°$
31. $\sin\frac{\pi}{2}$
32. $\cos\frac{4\pi}{3}$
33. $\cos\frac{3\pi}{4}$
34. $\sin\frac{4\pi}{3}$
35. $\cos\frac{11\pi}{6}$
36. $\sin 2\pi$



For the indicated point, tell if the value for $\sin\theta$ or $\cos\theta$ is positive, negative, or neither.

37. $\cos G$ 38. $\sin C$

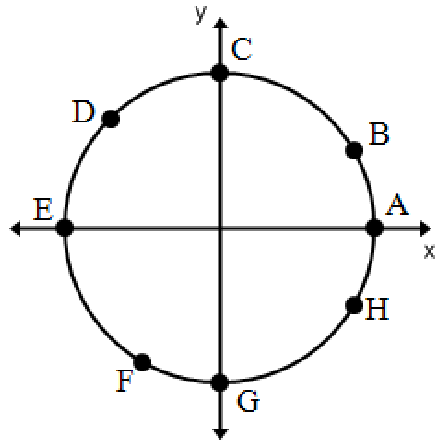
39. $\sin H$ 40. $\cos D$

41. $\cos B$ 42. $\sin E$

43. $\sin B$ 44. $\sin F$

45. $\sin G$ 46. $\cos A$

47. $\cos E$ 48. $\cos C$



HONORS

Previously we defined the six trigonometric functions. Notice that cosecant, secant, and cotangent are reciprocals of sine, cosine, and tangent, respectively.

The Six Trigonometric Functions

sine $(\theta) = \sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

cosine $(\theta) = \cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

tangent $(\theta) = \tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

cosecant $(\theta) = \csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

secant $(\theta) = \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

cotangent $(\theta) = \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\text{adjacent}}{\text{opposite}}$

Example 5:

a. $\sec\dfrac{\pi}{3}$

b. $\cot 330^\circ$

Answer:

a. $\cos\dfrac{\pi}{3} = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{1}{2}$ and secant is the reciprocal of cosine therefore,

$\sec\dfrac{\pi}{3} = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{2}{1} = 2$.

b. $\tan 330^\circ = \dfrac{\sin 330^\circ}{\cos 330^\circ} = -\dfrac{\sqrt{3}}{3}$ and cotangent is the reciprocal of tangent therefore,

$\cot 330^\circ = \dfrac{\cos 330^\circ}{\sin 330^\circ} = -\dfrac{3}{\sqrt{3}} = -\dfrac{3}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{3\sqrt{3}}{3} = -\sqrt{3}$.

Practice Exercises D

Use the unit circle to find the exact value.

1. $\cot\frac{11\pi}{6}$
2. $\csc\frac{\pi}{6}$
3. $\sec\frac{5\pi}{6}$
4. $\cot\frac{\pi}{2}$
5. $\csc\frac{7\pi}{6}$
6. $\sec\frac{7\pi}{4}$
7. $\cot 210°$
8. $\csc 225°$
9. $\sec 240°$
10. $\cot 45°$
11. $\csc 120°$
12. $\sec 60°$

Unit 3 Clusters 2 & 3 (F.TF.2 and F.TF.5): Graphing Sine and Cosine

Cluster 2: Extending the domain of trigonometric functions using the unit circle

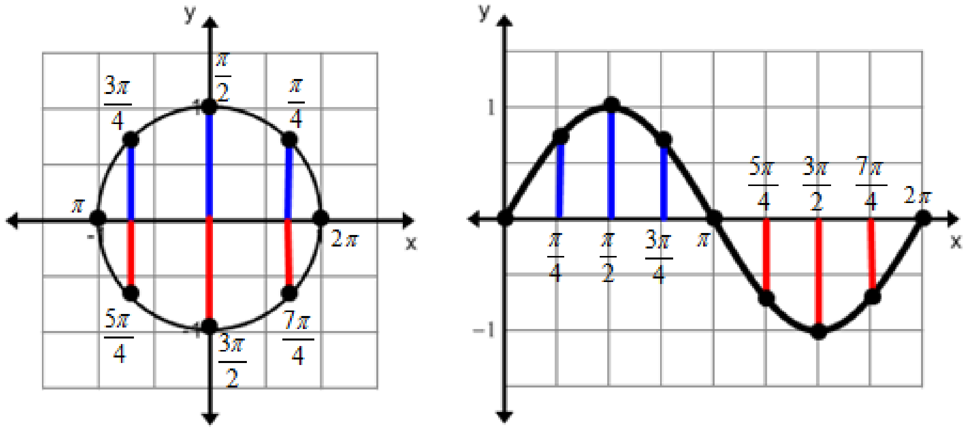
3.2 Explain how the unit circle enables the extension of trigonometric functions to all real numbers.

Cluster 3: Model periodic phenomena with trigonometric functions

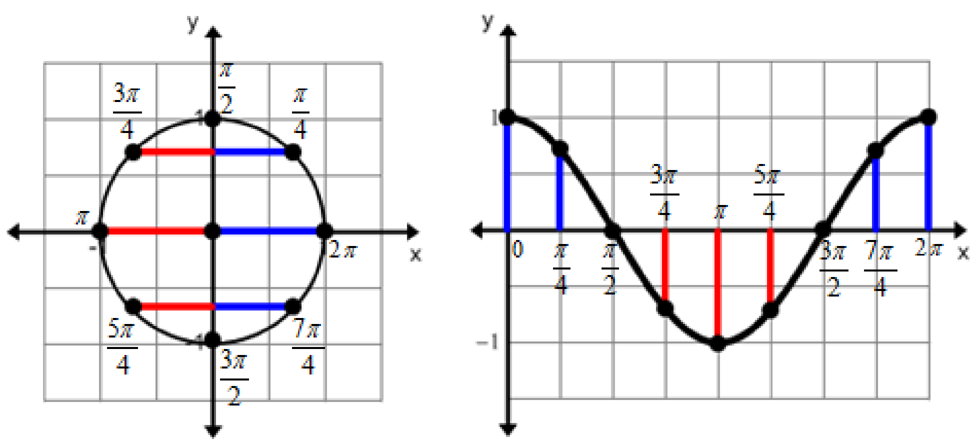
3.3 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Graphing Sine and Cosine

Plotting every angle and its corresponding sine value, which is the *y*-coordinate, for different angles on the unit circle, allows us to create the sine function where $x=\theta$ and $y=\sin\theta$.



Plotting every angle and its corresponding cosine value, which is the *x*-coordinate, for different angles on the unit circle, allows us to create the cosine function where $x=\theta$ and $y=\cos\theta$.



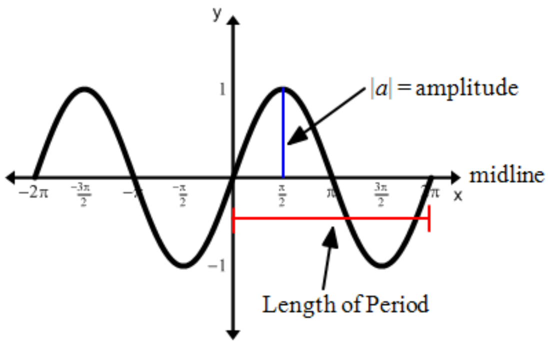
Keep in mind that we can extend this concept to all real numbers because rotating around the circle multiple times maps the new angle on to an existing coterminal angle. This type of function is called a **periodic** function because it has a pattern that repeats.

General Equations and Graphs

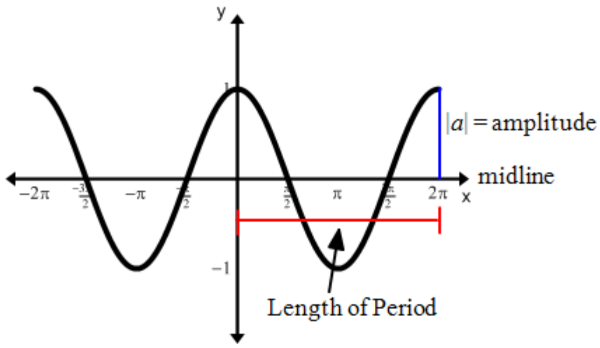
$$f(x) = a\sin(bx) + k \qquad \text{and} \qquad f(x) = a\cos(bx) + k$$

The domain of each function is the set of all real numbers, $(-\infty, \infty)$. The range of each function is $[-1, 1]$.

$f(x) = \sin(x)$



$f(x) = \cos(x)$



VOCABULARY

The **amplitude**, $|a|$, is half the difference between the maximum and minimum values of the function.

The **period**, $\dfrac{2\pi}{|b|}$, is the interval length needed to complete one cycle.

The **frequency**, $\dfrac{|b|}{2\pi}$, is the number of complete cycles a periodic function makes in a specific interval.

The **midline**, k, is the horizontal line that cuts the trigonometric function in half.

Example 1:
Identify the amplitude and period, then sketch one period of the graph.

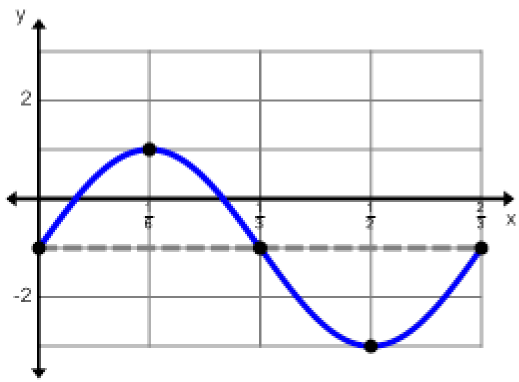
a. $f(x) = 2\sin(3\pi x) - 1$

b. $f(x) = -5\cos\left(\dfrac{1}{2}x\right) + 3$

a. $f(x) = 2\sin(3\pi x) - 1$

Amplitude $= |2| = 2$

Period $= \dfrac{2\pi}{|3\pi|} = \dfrac{2}{3}$



Draw the midline located at $y = -1$. Label the period $\frac{2}{3}$ and then divide that section into four even segments.

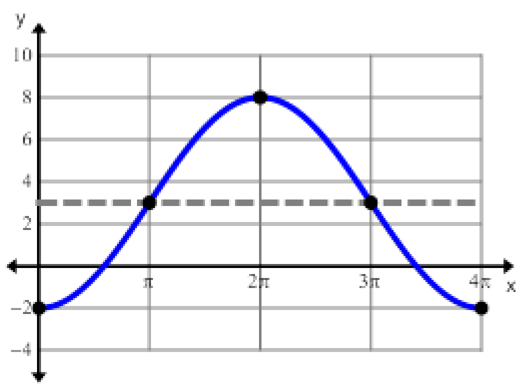
Determine the maximum, $y = k + a = -1 + 2 = 1$, and minimum, $y = k - a = -1 - 2 = -3$, values.

The sine function starts at the midline, rises to the maximum, decreases back to the midline, decreases to the minimum, and finally increases back to the midline. Plot these five points (midline, maximum, midline, minimum, midline) and connect the points.

b. $f(x) = -5\cos\left(\frac{1}{2}x\right) + 3$

Amplitude $= |-5| = 5$

Period $= \dfrac{2\pi}{\left|\frac{1}{2}\right|} = 4\pi$



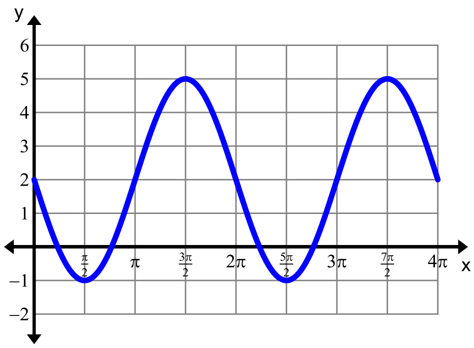
Draw the midline located at $y = 3$. Label the period 4π and then divide that section into four even segments.

Determine the maximum, $y = k + a = 3 + 5 = 8$, and minimum, $y = k - a = 3 - 5 = -2$, values.

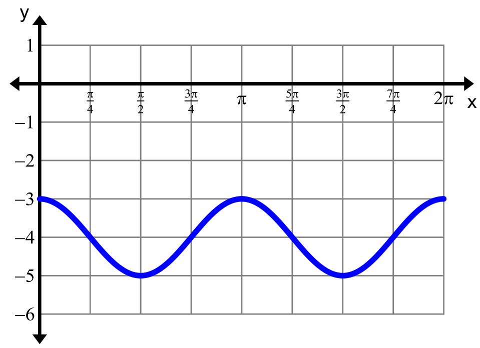
Ordinarily the cosine function starts at its maximum value, decreases to the midline, decreases to the minimum, increases to the midline, and increases to the maximum. This function has been reflected over the *x*-axis so plot the minimum, midline, maximum, midline, and then the minimum again and connect the points.

Example 2:
Identify the amplitude and period, and determine where the midline is located. Then write the equation for the function.

a. Use $f(x) = \sin x$ for your parent graph.



b. Use $f(x) = \cos x$ for your parent graph.



a. $y = 2$	Determine the midline.
Amplitude $= \dfrac{5-(-1)}{2} = 3$	Find the difference between the maximum and minimum values and divide by 2. Maximum Value 5 Minimum Value -1
Period $= 2\pi$	Notice the graph goes from its minimum to its maximum and back to the midline in the length of 2π.
$f(x) = -3\sin x + 2$	$2\pi = \dfrac{2\pi}{\lvert b \rvert}$ $\lvert b \rvert = \dfrac{2\pi}{2\pi}$ $b = 1$ The sine function usually starts at the midline and goes to its maximum, but this graph starts at the midline and goes to the minimum so the function has been reflected.

b. $y = -4$	Determine the midline
Amplitude $= \dfrac{-3-(-5)}{2} = 1$	Find the difference between the maximum and minimum values and divide by 2. Maximum Value -3 Minimum Value -5
Period $= \pi$	Notice that the graph goes from its minimum to its maximum and back to the midline in the length of π.
$f(x) = \cos 2x - 4$	$\pi = \dfrac{2\pi}{\lvert b \rvert}$ $\lvert b \rvert = \dfrac{2\pi}{\pi}$ $b = 2$

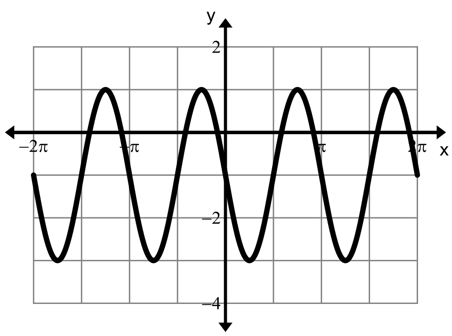
Practice Exercises A

Identify the amplitude and period. Then sketch one period of the graph.

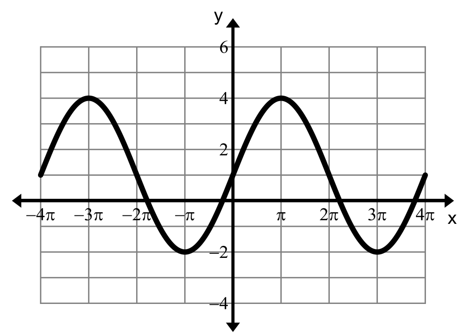
1. $f(x) = \sin x + 2$
2. $f(x) = \sin x - 2$
3. $f(x) = \cos x + 3$
4. $f(x) = \cos x - 3$
5. $f(x) = 2\sin\left(\frac{1}{2}x\right) + 1$
6. $f(x) = 2\cos\left(\frac{1}{2}x\right) - 1$
7. $f(x) = -\sin(\pi x)$
8. $f(x) = -3\cos(2\pi x)$
9. $f(x) = 4\sin(2x) - 2$
10. $f(x) = 4\cos\left(\frac{2}{3}x\right) + 3$
11. $f(x) = -\dfrac{1}{2}\sin x + 3$
12. $f(x) = 2\cos x - 4$

Identify the amplitude and period, and determine where the midline is located. Then write the equation for the function. Use $f(x) = \sin(x)$ for the parent function.

13.



14.



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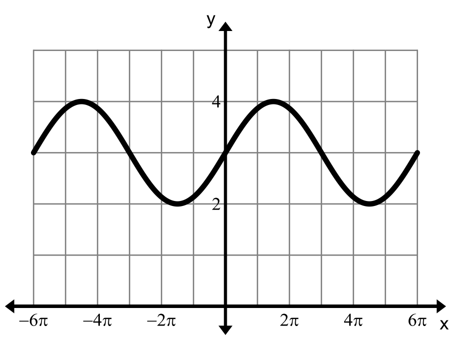

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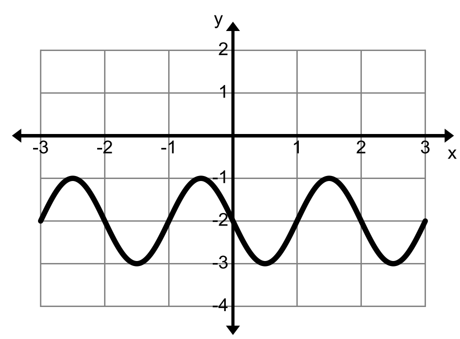
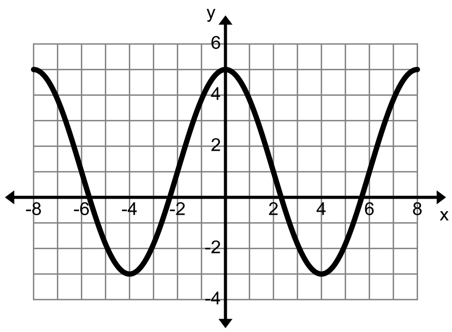


Identify the amplitude and period, and determine where the midline is located. Then write the equation for the function. Use $f\left(x\right)=\cos\left(x\right)$ for the parent function.

17.



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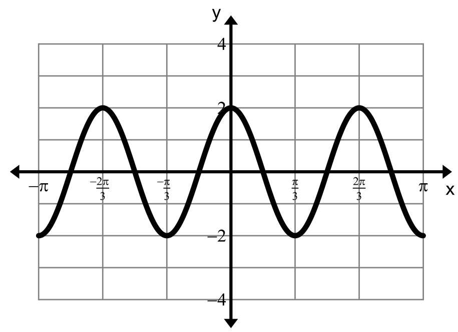

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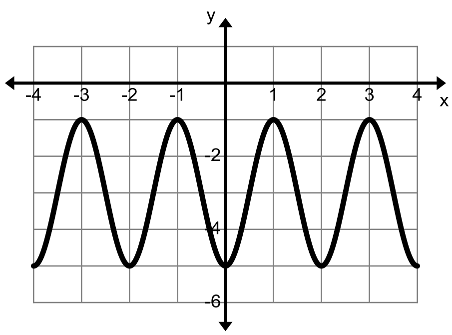

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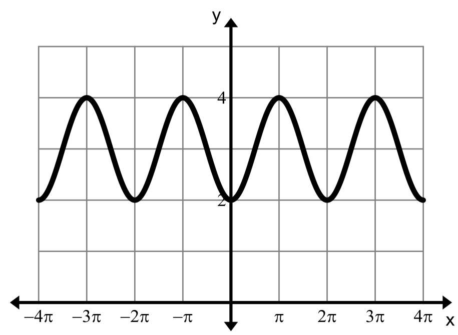
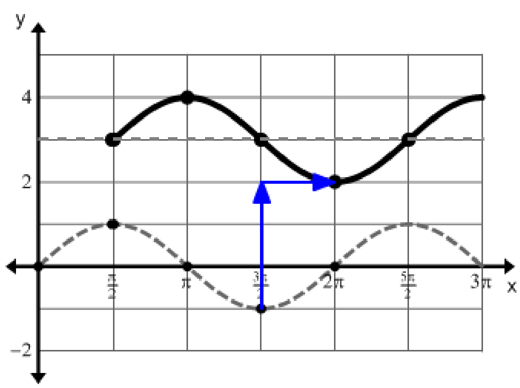


Horizontal Shift (Phase Shift)

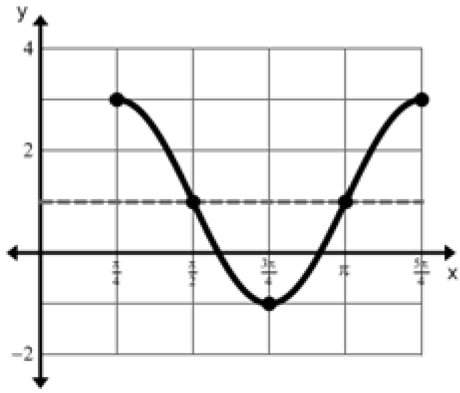
VOCABULARY When a horizontal shift is performed on a trigonometric function it is called a **phase shift**. The general equations are $f(x)=a\sin\left[b(x-h)\right]+k$ or $f(x)=a\cos\left[b(x-h)\right]+k$, where h is the number of units the graph is shifted horizontally.

Example 3:
Identify the amplitude, period, phase shift, and vertical shift, then sketch one period of the graph.

a. $f(x)=\sin\left(x-\dfrac{\pi}{2}\right)+3$ b. $f(x)=2\cos\left(2\left(x-\dfrac{\pi}{4}\right)\right)+1$ c. $f(x)=-\cos\left(\frac{1}{3}x+\pi\right)-2$

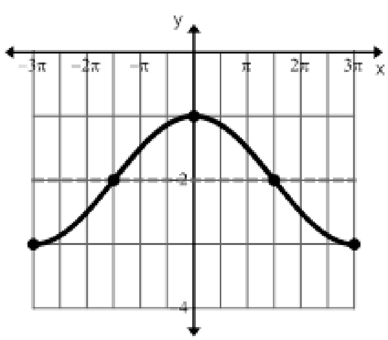
a. $f(x)=\sin\left(x-\dfrac{\pi}{2}\right)+3$ Amplitude: 1 Period: $\dfrac{2\pi}{\lvert 1\rvert}=2\pi$ Phase shift: right $\dfrac{\pi}{2}$ units Vertical shift: up 3 units	The amplitude is $\lvert a\rvert$. The period is $\dfrac{2\pi}{\lvert b\rvert}$. $h=\dfrac{\pi}{2}$ $k=3$
	Each point has been shifted up 3 units and to the right $\dfrac{\pi}{2}$ units. Follow the pattern of midline, max, midline, min, and midline to plot the shifted sine function.

b. $f(x)=2\cos\left(2\left(x-\dfrac{\pi}{4}\right)\right)+1$ Amplitude: 2 Period: $\dfrac{2\pi}{\lvert 2\rvert}=\pi$ Phase shift: right $\dfrac{\pi}{4}$ units Vertical shift: up 1 unit	The amplitude is $\lvert a\rvert$. The period is $\dfrac{2\pi}{\lvert b\rvert}$. $h=\dfrac{\pi}{4}$ $k=1$



The first point has been shifted $\frac{\pi}{4}$ units to the right and up 1 unit (amplitude of 2). Follow the pattern of max, midline, min, midline, and max to plot the rest of the shifted cosine function.

c. $f(x)=-\cos\left(\frac{1}{3}x+\pi\right)-2$ $f(x)=-\cos\left[\frac{1}{3}(x+3\pi)\right]-2$ Amplitude: 1 Period: $\frac{2\pi}{\left\lvert\frac{1}{3}\right\rvert}=6\pi$ Phase shift: right 3π units Vertical shift: down 2 units	Rewrite the function so that the coefficient of *x* has been factored out of both terms. The amplitude is $\lvert a\rvert$. The period is $\frac{2\pi}{\lvert b\rvert}$. $h=3\pi$ $k=2$



The first point has been shifted 3π units to the left and down 2 units. The function has been reflected across the *x*-axis. Follow the pattern of min, midline, max, midline, and min to plot the rest of the shifted cosine function.

Practice Exercises B

Identify the amplitude, period, phase shift, and vertical shift, then sketch one period of the graph.

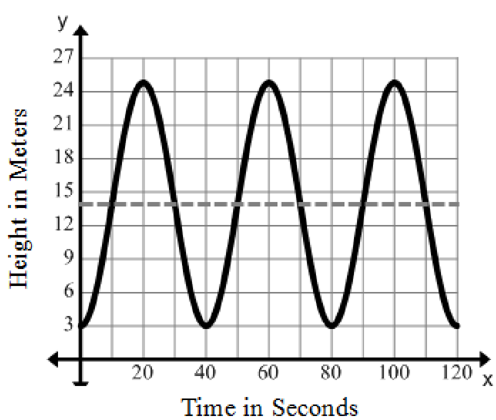
1. $f(x)=\sin\left(x-\frac{\pi}{2}\right)+2$
2. $f(x)=\cos(x+\pi)-1$
3. $f(x)=-2\sin(x-\pi)$
4. $f(x)=-\cos\left[\frac{1}{2}\left(x+\frac{\pi}{2}\right)\right]$
5. $f(x)=3\sin\left[2\left(x-\frac{3\pi}{2}\right)\right]$
6. $f(x)=\cos\left[3\left(x+\frac{3\pi}{2}\right)\right]+1$
7. $f(x)=\sin\left(3x-\frac{3\pi}{2}\right)+4$
8. $f(x)=-3\cos(2x+\pi)$
9. $f(x)=\sin\left(\frac{3}{2}x+\frac{3\pi}{4}\right)-3$

Modeling Periodic Phenomena

Example 4:

The Ferris wheel at Lagoon has a diameter of 21.8 meters. It rotates on a platform that is 3 meters above the ground. The Ferris wheel completes one revolution in 40 seconds. Write an equation to model the situation. Then sketch a graph of height versus time, extending the graph for more than one revolution.

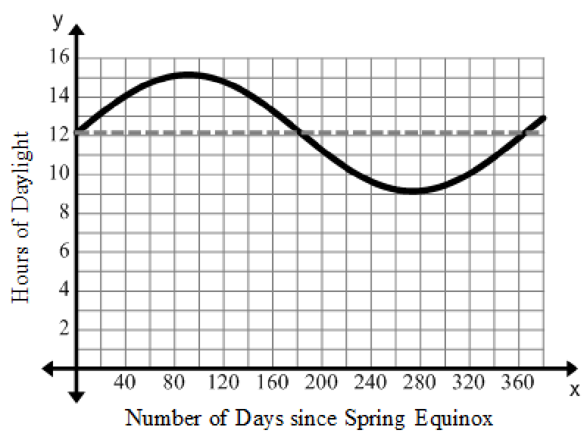
$\lvert a\rvert = \frac{24.8-3}{2} = \frac{21.8}{2} = 10.9$	The amplitude is half the difference of the maximum and minimum values. Minimum: 3 meters Maximum: 3 + 21.8 = 24.8 meters
$40 = \frac{2\pi}{b}$ $b = \frac{2\pi}{40} = \frac{\pi}{20}$	The period is 40 seconds. Find the value of *b*.
$k = 10.9 + 3 = 13.9$	The center of the Ferris wheel is 10.9 meters above the platform. The value of *k* is the distance the center of the Ferris wheel is from the ground.
$f(x) = -10.9\cos\left(\frac{\pi}{20}x\right) + 13.9$	In order to get on the Ferris wheel, the cart must be at its minimum value. Cosine usually starts at its maximum, but if you reflect it across the *x*-axis, then it will start at its minimum.



Example 5:

In Salt Lake City, Utah, at the spring equinox (March 20, 2013) there were 12 hours and 9 minutes of daylight. The longest day (June 20, 2013) and shortest day (December 21, 2013) of the year vary from the equinox by approximately 3 hours. Write a sine function that relates the number of days to the variation of daylight hours in Salt Lake City. Graph the model, showing at least one year.

$\lvert a \rvert = 3$	The amount of daylight varies from the equinox by 3 hours so the amplitude is 3.
$365 = \frac{2\pi}{b}$ $b = \frac{2\pi}{365}$	The period is 365 days. Find the value of *b*.
$k = 12 + \frac{9}{60} = 12.15$	The midline is at 12 hours and 9 minutes because the amount of daylight is ± 3 hours from this value.
$f(x) = 3\sin\left(\frac{2\pi}{365}x\right) + 12.15$	Substitute in known values.



Practice Exercises C

1. A buoy oscillates up and down as waves go past. The buoy moves a total of 3.6 feet from its low point to its high point, and then returns to its high point every 8 seconds. Write a cosine function modeling the buoy's vertical position at any time *t*.

2. A Ferris wheel 50 feet in diameter makes one revolution every 40 seconds. The center of the wheel is 30 feet above the ground. Write a cosine function to model the height of a car on the Ferris wheel at any time *t*.

3. Low tide is at 10:15 am and high tide is at 4:15 pm. The water level varies 64 inches between low and high tide. Write a cosine function to represent the change in water level.

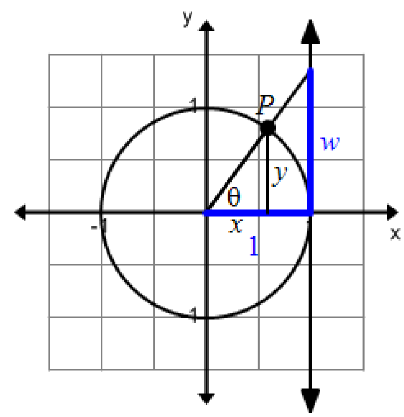
4. The lowest pitch a human can easily hear has a frequency of 30 cycles per second. Write a sine function representing the sound wave of the pitch. (Amplitude is 1)

5. The highest pitch a human can easily hear has a frequency of 20,000 cycles per second. Write a sine function representing the sound wave of the pitch. (Amplitude is 1)

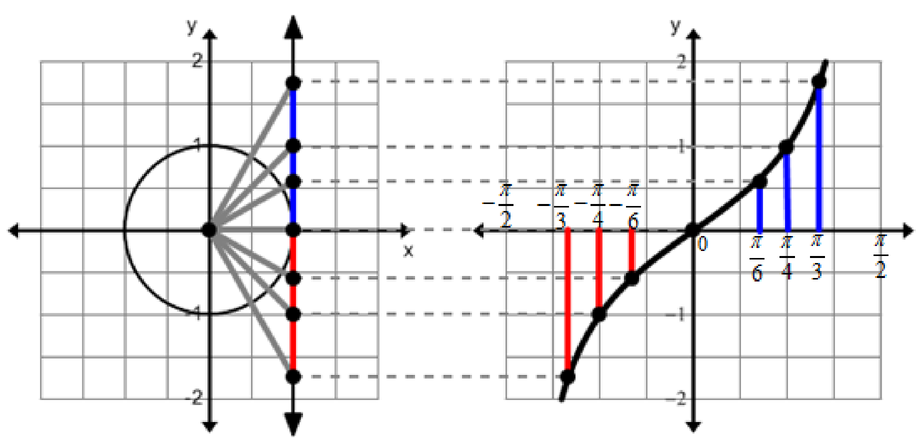
6. In Buenos Aires, Argentina, the average monthly temperature is the highest in January and the lowest in July. It ranges from $76°$ F to $51°$ F. Write a cosine function that models the change in temperature according to the month of the year.

Graphing Tangent (Honors)

In order to graph the tangent function, $f(x) = \tan x$, we have to examine the relationship between two similar triangles drawn at the right. The smaller triangle is inscribed in the unit circle and intersects the circle at the point $P(x, y)$. The larger triangle shares a vertex with the smaller triangle, but its shorter leg is the length of the radius of the circle. Setting up the proportion we get: $\frac{\text{long leg}}{\text{short leg}} = \frac{y}{x} = \frac{w}{1}$. Simplifying this statement we can see that $w = \frac{y}{x}$. Remember that $y = \sin\theta$, $x = \cos\theta$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$. Thus, the length of the longer leg of the triangle is equal to $w = \frac{y}{x} = \tan\theta$.



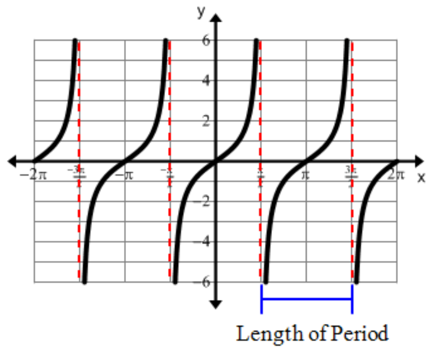
If we draw more similar triangles for different angles on the unit circle, then we would be able to plot the graph of the tangent function where $x = \theta$ and $y = \tan\theta$. Recall that tangent is undefined when $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$. The graph would have vertical asymptotes at those values.



General Equation and Graph

$$f(x) = a\tan(bx) + k$$

The domain of the function is all real numbers except odd multiples of $\frac{\pi}{2}$. The range of each function is the set of all real numbers. The graph crosses the *y*-axis half way between the two asymptotes.



VOCABULARY

Period, $\frac{\pi}{|b|}$, is the interval length needed to complete one cycle.

Frequency, $\frac{|b|}{\pi}$, is the number of complete cycles a periodic function makes in a specific interval.

The **Asymptotes** are determined using the following:

The first set, centered around the origin is given by $x = 0 \pm \frac{\pi}{2|b|}$. To determine the remaining asymptotes, add the period to the previous asymptote.

Example 6:
Identify the period, vertical asymptotes, and *y*-intercept, then sketch one period of the graph.

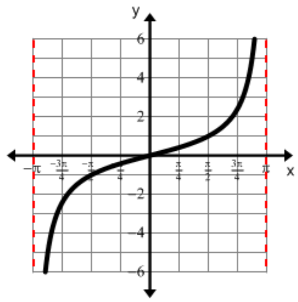
a. $f(x) = \tan\left(\frac{1}{2}x\right)$

b. $f(x) = \tan(\pi x) + 2$

<table>
<tr><td>a. Period $= \frac{\pi}{|b|} = \frac{\pi}{\left|\frac{1}{2}\right|} = 2\pi$</td><td rowspan="2">Draw the asymptotes.

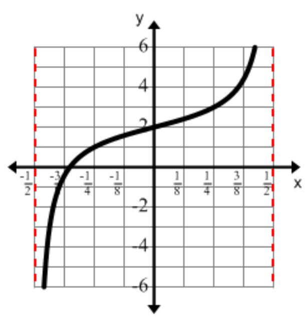
The x-coordinate of the y-intercept is located half way between $\pm\pi$ and the y-coordinate is the vertical shift.

Sketch the graph.</td></tr>
<tr><td>Asymptotes
$x = 0 \pm \frac{\pi}{2|b|}$
$= 0 \pm \frac{\pi}{2\left|\frac{1}{2}\right|}$
$= \pm\pi$</td></tr>
</table>

<table>
<tr><td>The y-intercept is at $(0,0)$</td><td rowspan="2"></td></tr>
<tr><td>
</td></tr>
</table>

<table>
<tr><td>b. Period $= \dfrac{\pi}{|b|} = \dfrac{\pi}{|\pi|} = 1$</td><td rowspan="4">Draw the asymptotes.

The x-coordinate of the y-intercept is located half way between $\pm\dfrac{1}{2}$ and the y-coordinate is the vertical shift.

Sketch the graph.</td></tr>
<tr><td>Asymptotes
$x = 0 \pm \dfrac{\pi}{2|b|}$
$= 0 \pm \dfrac{\pi}{2|\pi|}$
$= \pm\dfrac{1}{2}$</td></tr>
<tr><td>The y-intercept is at $(0,2)$</td></tr>
<tr><td>
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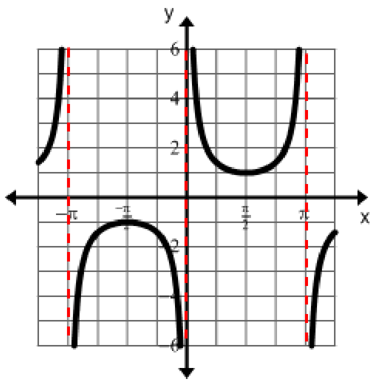
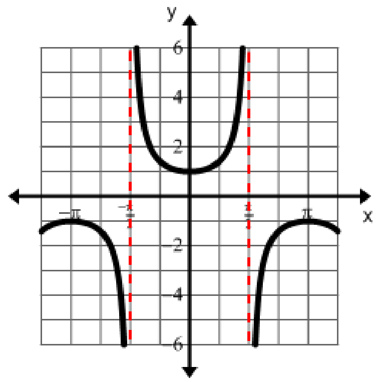
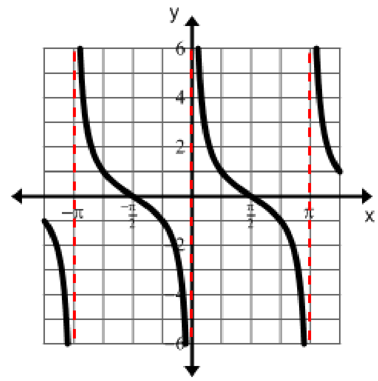
Practice Exercises D

Identify the period, vertical asymptotes, and y-intercept, then sketch one period of the graph.

1. $f(x) = \tan\left(\dfrac{x}{4}\right) + 3$
2. $f(x) = \tan(2x) + 1$
3. $f(x) = \tan x - 4$
4. $f(x) = -\tan\left(\tfrac{1}{2}x\right)$
5. $f(x) = -2\tan(2x)$
6. $f(x) = 5\tan(x) - 3$
7. $f(x) = 3\tan(2x) - 2$
8. $f(x) = -\tan(\pi x) + 4$
9. $f(x) = \tfrac{1}{2}\tan\left(\dfrac{\pi}{2}x\right)$

Reciprocal Trigonometric Functions (Honors)

Graphs of the Reciprocal Trigonometric Functions		
$f(x)=\csc x$	$f(x)=\sec x$	$f(x)=\cot x$
		

Summary of the Basic Trigonometric Functions						
Function	Period	Domain	Range	Asymptotes	Zeros	Even/Odd
$f(x)=\sin x$	2π	$(-\infty,\infty)$	$[-1,1]$	None	$n\pi$	Odd
$f(x)=\cos x$	2π	$(-\infty,\infty)$	$[-1,1]$	None	$\frac{\pi}{2}+n\pi$	Even
$f(x)=\tan x$	π	$x\neq\frac{\pi}{2}+n\pi$	$(-\infty,\infty)$	$x=\frac{\pi}{2}+n\pi$	$n\pi$	Odd
$f(x)=\csc x$	2π	$x\neq n\pi$	$(-\infty,-1]\cup[1,\infty)$	$x=n\pi$	None	Odd
$f(x)=\sec x$	2π	$x\neq\frac{\pi}{2}+n\pi$	$(-\infty,-1]\cup[1,\infty)$	$x=\frac{\pi}{2}+n\pi$	None	Even
$f(x)=\cot x$	π	$x\neq n\pi$	$(-\infty,\infty)$	$x=n\pi$	$\frac{\pi}{2}+n\pi$	Odd

Unit 3 Clusters 2 and 3 HONORS (F.TF.4): Symmetry and Periodicity

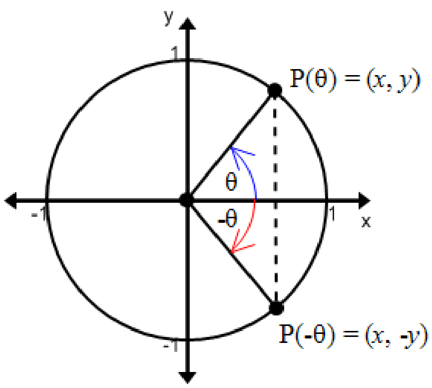
Cluster 2: Extending the domain of trigonometric functions using the unit circle

3.2 Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

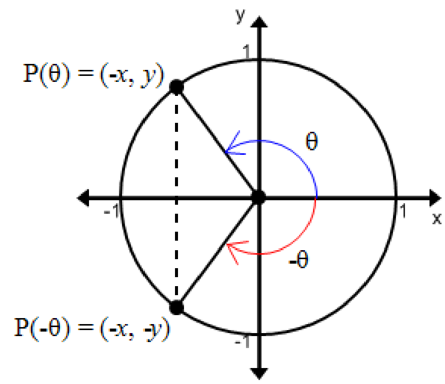
Odd and Even Symmetry

Symmetry in a unit circle shows that for any real number θ, the points $P(\theta)$ and $P(-\theta)$, where $x=\cos\theta$ and $y=\sin\theta$, located on the terminal side of an angle θ will have the same cosine values and opposite sine values. This holds true regardless of which quadrant contains the terminal side of the angle.

1st and 4th Quadrants



2nd and 3rd Quadrants



Therefore, $\cos(-\theta)=\cos(\theta)$ making cosine an even function and $\sin(-\theta)=-\sin(\theta)$ making sine an odd function.

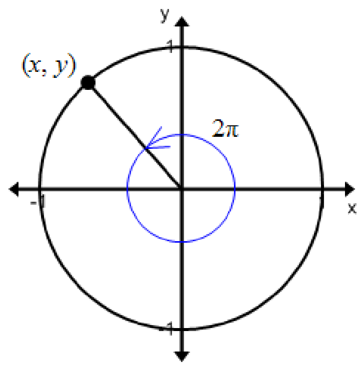
Example 1:

Using symmetry, find exact values of $\sin\theta$ and $\cos\theta$ if $\theta=-\frac{\pi}{3}$.

$\sin(-\theta)=-\sin(\theta)$ $\sin\left(-\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$	Sine is an odd function.
$\cos(-\theta)=\cos(\theta)$ $\cos\left(-\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$	Cosine is an even function.

Periodicity

A function, *f*, is said to be periodic if there is a positive number *P* such that $f(\theta+P)=f(\theta)$ for all θ in the domain. The smallest number *P* for which this occurs is called the period of *f*.   Sine and cosine are periodic functions and have a period of 2π. Any point (x,y) on the unit circle, will be repeated after a rotation of $\pm 2\pi$. Therefore, $(x\pm 2\pi, y\pm 2\pi)$ will be mapped on to (x,y). $\sin(\theta\pm 2n\pi)=\sin(\theta)$ and $\cos(\theta\pm 2n\pi)=\cos(\theta)$, where *n* is any integer.



Example 2:

Using periodicity evaluate the following:

a. $\sin(9\pi)$ b. $\cos\left(\frac{31\pi}{6}\right)$

a. $\sin(9\pi)$	
$\sin(9\pi)=\sin(4(2\pi)+\pi)$	It is obvious that the angle 9π is more than one rotation of the unit circle. It is in fact 4 full rotations and π more.
$\sin(9\pi)=\sin(\pi)=0$	Because of periodicity this statement is true.

b. $\cos\left(\frac{31\pi}{6}\right)$	
$\cos\left(\frac{31\pi}{6}\right)=\cos\left(2(2\pi)+\frac{7\pi}{6}\right)$	One complete rotation of the unit circle in terms of sixths would be $2\pi=\frac{12\pi}{6}$. $\frac{31\pi}{6}$ is at least twice $\left(\frac{24\pi}{6}\right)$ around the unit circle, but not three $\left(\frac{36\pi}{6}\right)$ times around it.
$\cos\left(\frac{31\pi}{6}\right)=\cos\left(\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2}$	Because of periodicity this statement is true.

Practice Exercises A

Use symmetry to find exact values of $\sin\theta$ and $\cos\theta$ for the given angle.

1. $\theta = -\frac{2\pi}{3}$
2. $\theta = -\frac{\pi}{6}$
3. $\theta = -\frac{3\pi}{4}$
4. $\theta = -\frac{5\pi}{3}$
5. $\theta = -\frac{5\pi}{6}$
6. $\theta = -\frac{\pi}{2}$

Use periodicity to evaluate the expression.

7. $\sin\left(\frac{13\pi}{6}\right)$
8. $\cos\left(\frac{14\pi}{3}\right)$
9. $\sin\left(\frac{13\pi}{2}\right)$
10. $\cos\left(\frac{9\pi}{4}\right)$
11. $\sin\left(\frac{61\pi}{3}\right)$
12. $\cos\left(\frac{23\pi}{6}\right)$
13. $\sin\left(\frac{27\pi}{4}\right)$
14. $\cos(20\pi)$
15. $\sin\left(\frac{7\pi}{2}\right)$

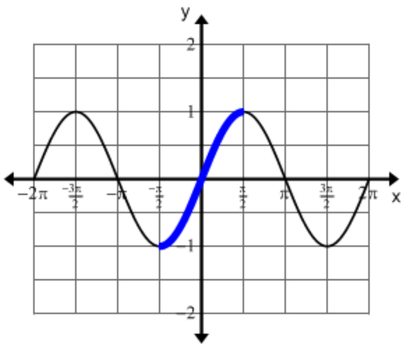
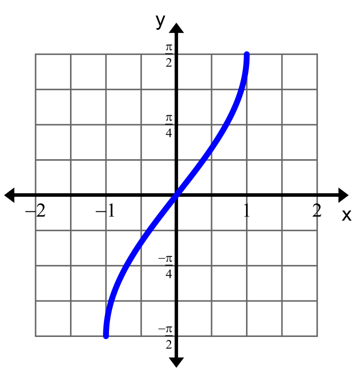
Unit 3 Clusters 2 and 3 HONORS (F.TF.6 and F.TF.7): Inverse Trigonometric Functions

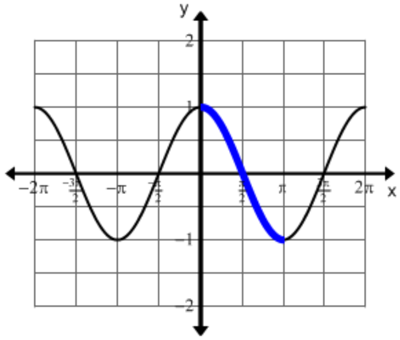
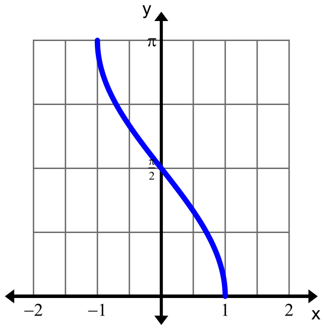
Cluster 2: Extending the domain of trigonometric functions using the unit circle

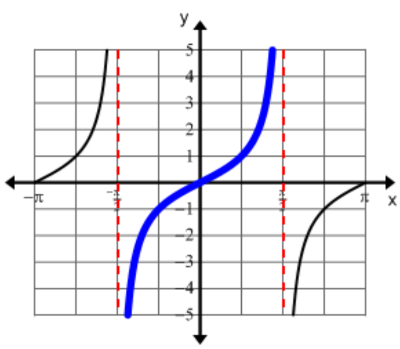
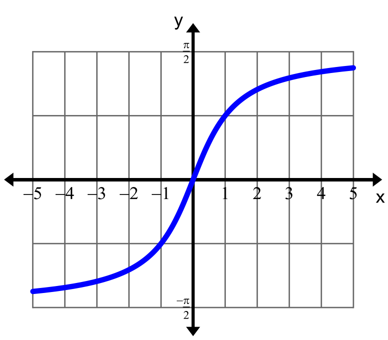
- 3.3H Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
- 3.3H Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology and interpret them in terms of the context.

Recall that in order for the inverse of a function to be a function, the original function must be a one-to-one function and meet the criteria for the vertical and horizontal line tests. Not all functions meet the criteria to have an inverse which is also a function. However, if the domain is restricted so that the original function is one-to-one, then the inverse will be a function. Determining which portion of the domain to use can be challenging. Generally, it is best to find an interval of the domain where the function is always increasing or always decreasing.

Graphing Inverse Trigonometric Functions

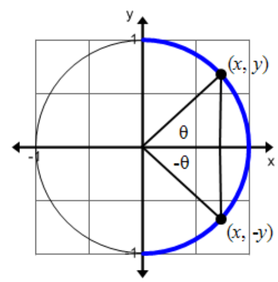
$f(x)=\sin(x)$ 	The sine function does not pass the horizontal line test. In order to graph its inverse, the domain will have to be restricted. The function is increasing on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. By restricting the domain to this interval, it will allow its inverse to be constructed.
$f(x)=\sin^{-1}(x)$ 	The inverse of the sine function (**arcsine function**) is denoted two ways: $f(x)=\sin^{-1}(x)$ or $f(x)=\arcsin(x)$. The x-coordinate now represents the sine value and the y-coordinate represents the unique angle. The domain is $[-1,1]$ and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

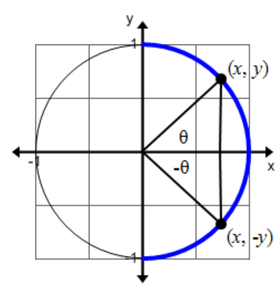
$f(x)=\cos(x)$ 	The cosine function does not pass the horizontal line test. In order to graph its inverse, the domain will have to be restricted. The function is decreasing on the interval $(0,\pi)$. By restricting the domain to this interval, it will allow its inverse to be constructed.
$f(x)=\cos^{-1}(x)$ 	The inverse of the cosine function (**arccosine function**) is denoted two ways: $f(x)=\cos^{-1}(x)$ or $f(x)=\arccos(x)$. The *x*-coordinate now represents the cosine value and the *y*-coordinate represents the unique angle. The domain is $[-1,1]$ and the range is $[0,\pi]$.

$f(x)=\tan(x)$ 	The tangent function does not pass the horizontal line test. In order to graph its inverse, the domain will have to be restricted. The function is increasing on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. By restricting the domain to this interval, it will allow its inverse to be constructed.
$f(x)=\tan^{-1}(x)$ 	The inverse of the tangent function (**arctangent function**) is denoted two ways: $f(x)=\tan^{-1}(x)$ or $f(x)=\arctan(x)$. The *x*-coordinate now represents the tangent value and the *y*-coordinate represents the unique angle. The domain is $(-\infty,\infty)$ and the range is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Example 1:
Find the exact value of the expression without a calculator.

a. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ b. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

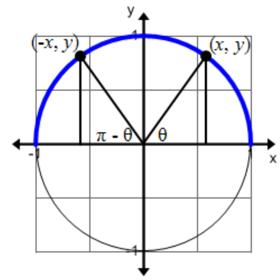
a. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ $\sin\theta = \frac{\sqrt{2}}{2}$	This is really asking us to find the angle, θ, that is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine value is $\frac{\sqrt{2}}{2}$.
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$	The inverse sine function is restricted to the first and fourth quadrants of the unit circle. Sine is positive in the first quadrant, therefore, θ will be an angle from the first quadrant. 

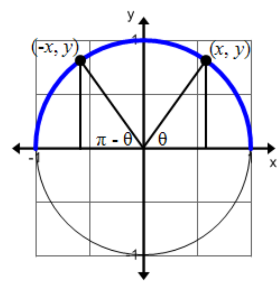
b. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\sin\theta = -\frac{\sqrt{3}}{2}$	This is really asking us to find the angle, θ, that is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine value is $-\frac{\sqrt{3}}{2}$.
$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$	The inverse sine function is restricted to the first and fourth quadrants of the unit circle. Sine is negative in the fourth quadrant, therefore, θ will be an angle from the fourth quadrant. 

Example 2:
Find the exact value of the expression without a calculator.

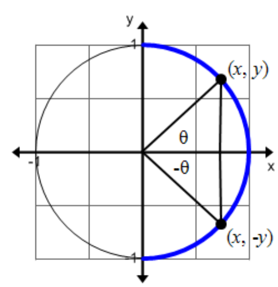
a. $\cos^{-1}\left(\frac{1}{2}\right)$ b. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

a. $\cos^{-1}\left(\frac{1}{2}\right)$ $\cos\theta = \frac{1}{2}$	This is really asking us to find the angle, θ, that is in the interval $[0, \pi]$ whose cosine value is $\frac{1}{2}$.

$\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$	The inverse cosine function is restricted to the first and second quadrants of the unit circle. Cosine is positive in the first quadrant, therefore, θ will be an angle from the first quadrant. 

b. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\cos\theta=-\frac{\sqrt{2}}{2}$	This is really asking us to find the angle, θ, that is in the interval $[0,\pi]$ whose cosine value is $-\frac{\sqrt{2}}{2}$.
$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}$	The inverse cosine function is restricted to the first and second quadrants of the unit circle. Cosine is negative in the second quadrant, therefore, θ will be an angle from the second quadrant. 

Example 3:
Find the exact value of the expression $\tan^{-1}(1)$ without using a calculator.

a. $\tan^{-1}(1)$ $\tan\theta=1$	This is really asking us to find the angle, θ, that is in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent value is 1.
$\tan^{-1}(1)=\frac{\pi}{4}$	The inverse tangent function is restricted to the first and fourth quadrants of the unit circle. Tangent is positive in the first quadrant, therefore, θ will be an angle from the first quadrant. 

Practice Exercises A

Find the exact value of the expression without a calculator.

1. $\sin^{-1}\left(\frac{1}{2}\right)$	2. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$	3. $\sin^{-1}(1)$	4. $\sin^{-1}\left(-\frac{1}{2}\right)$
5. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$	6. $\sin^{-1}(0)$	7. $\sin^{-1}(-1)$	8. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
9. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$	10. $\cos^{-1}(1)$	11. $\cos^{-1}(-1)$	12. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
13. $\cos^{-1}\left(-\frac{1}{2}\right)$	14. $\cos^{-1}(0)$	15. $\tan^{-1}\left(\sqrt{3}\right)$	16. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$
17. $\tan^{-1}(0)$	18. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$	19. $\tan^{-1}(-1)$	20. $\tan^{-1}\left(-\sqrt{3}\right)$

Compositions Involving Inverse Trigonometric Functions

You may recall that for inverse functions the following properties are true: $f\left(f^{-1}(x)\right)=x$ and $f^{-1}\left(f(x)\right)=x$. Thus, the following properties are true for trigonometric functions and their inverses:

- $\sin\left(\sin^{-1}(x)\right)=x$ for every x in the interval $[-1,1]$.
- $\sin^{-1}\left(\sin(x)\right)=x$ for every x in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
- $\cos\left(\cos^{-1}(x)\right)=x$ for every x in the interval $[-1,1]$.
- $\cos^{-1}\left(\cos(x)\right)=x$ for every x in the interval $[0,\pi]$.
- $\tan\left(\tan^{-1}(x)\right)=x$ for every x in the interval $(-\infty,\infty)$.
- $\tan^{-1}\left(\tan(x)\right)=x$ for every x in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Example 4:
Find the exact value of the expression.

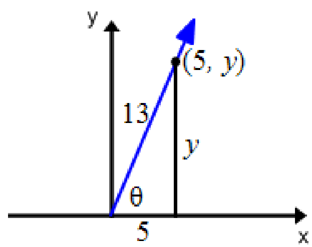
a. $\sin\left(\sin^{-1}(0.3)\right)$ b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$ c. $\tan^{-1}\left(\tan(\pi)\right)$

a. $\sin\left(\sin^{-1}(0.3)\right)$ $\sin\left(\sin^{-1}(0.3)\right)=0.3$	$x=0.3$ which is in the interval $[-1,1]$. Use the properties of inverse trigonometric functions.
b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$ $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$	$x=\frac{7\pi}{6}$ which is not in the interval $[0,\pi]$. Evaluate the inside function $\cos\left(\frac{7\pi}{6}\right)$. Find the angle whose cosine value is $-\frac{\sqrt{3}}{2}$.
c. $\tan^{-1}\left(\tan(\pi)\right)$ $\tan^{-1}(0)$ $\tan^{-1}(0)=0$	$x=\pi$ which is not in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Evaluate the inside function $\tan(\pi)$. Find the angle whose tangent value is 0.

Example 5:
Find the exact value of the expression.

a. $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$ b. $\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right)$

a. Using the definition of the inverse cosine function, we can rewrite the inside function as $\cos\theta=\frac{5}{13}$. The cosine value is positive, which means that the angle we are looking for is in the first quadrant, as the picture below demonstrates.

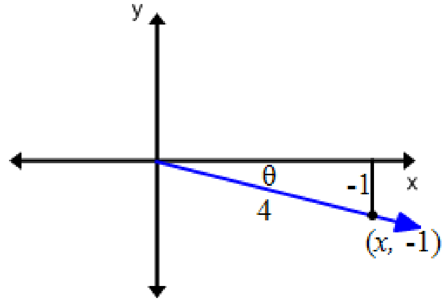


It is not necessary to find the actual value of the angle because $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$ is asking us to find the tangent of the angle whose cosine is $\frac{5}{13}$. Recall that $\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$.

Using the Pythagorean theorem, the opposite side is:
$y=\sqrt{13^2-5^2}=\sqrt{169-25}=\sqrt{144}=12$.

$$\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)=\tan(\theta)=\frac{12}{5}$$

b. Using the definition of the inverse sine function, we can rewrite the inside function as $\sin\theta = -\frac{1}{4}$. The sine value is negative, which means that the angle we are looking for is in the fourth quadrant, as the picture below demonstrates.



It is not necessary to find the actual value of the angle because $\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right)$ is asking us to find the cosine of the angle whose sine is $-\frac{1}{4}$. Recall that $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.

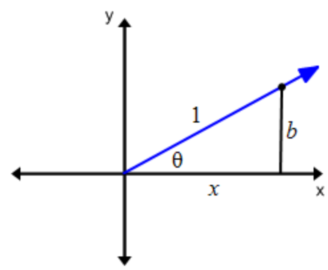
Using the Pythagorean theorem, the adjacent side is: $x = \sqrt{4^2 - (-1)^2} = \sqrt{16-1} = \sqrt{15}$.

$\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right) = \cos(\theta) = \frac{\sqrt{15}}{4}$.

Example 6:

Write $\sin\left(\cos^{-1} x\right)$ as an algebraic expression if $0 < x \le 1$.

The expression is limited to the first quadrant by the statement $0 < x \le 1$. Using the definition of the inverse cosine function, we can rewrite the inside function as $\cos\theta = x = \frac{x}{1}$.



It is not necessary to find the actual value of the angle because $\sin\left(\cos^{-1} x\right)$ is asking us to find the sine of the angle whose cosine is $\frac{x}{1}$. Recall that $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.

Using the Pythagorean theorem, the opposite side is: $b = \sqrt{1^2 - x^2}$.

$\sin\left(\cos^{-1} x\right) = \sin(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.

Practice Exercises B

Find the exact value of the expression.

1. $\sin\left(\sin^{-1}(0.8)\right)$
2. $\cos\left(\cos^{-1}(0.45)\right)$
3. $\tan\left(\tan^{-1}(7)\right)$
4. $\tan\left(\tan^{-1}(100)\right)$
5. $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$
6. $\cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$
7. $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$
8. $\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)$
9. $\sin^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$
10. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$
11. $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$
12. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
13. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$
14. $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$
15. $\cos^{-1}\left(\cos\left(\frac{11\pi}{6}\right)\right)$

Find the exact value of the expression.

16. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$
17. $\sin\left(\tan^{-1}(1)\right)$
18. $\tan\left(\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$
19. $\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$
20. $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$
21. $\cos\left(\tan^{-1}\left(\frac{7}{24}\right)\right)$
22. $\tan\left(\sin^{-1}\left(\frac{5}{13}\right)\right)$
23. $\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$
24. $\sin\left(\cos^{-1}\left(-\frac{4}{5}\right)\right)$
25. $\sin\left(\tan^{-1}\left(\frac{15}{8}\right)\right)$
26. $\cos\left(\sin^{-1}\left(\frac{4}{7}\right)\right)$
27. $\tan\left(\sin^{-1}\left(\frac{3}{4}\right)\right)$

Write each expression as an algebraic expression if $0 < x \le 1$.

28. $\tan\left(\sin^{-1}(x)\right)$
29. $\sin\left(\tan^{-1}(x)\right)$
30. $\sin\left(\cos^{-1}\left(\frac{1}{x}\right)\right)$
31. $\tan\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$
32. $\cos\left(\tan^{-1}\frac{x}{\sqrt{3}}\right)$
33. $\sin\left(\tan^{-1}\left(\frac{x}{2}\right)\right)$
34. $\cos\left(\sin^{-1}(2x)\right)$
35. $\cos\left(\sin^{-1}\left(\frac{\sqrt{x^2-9}}{x}\right)\right)$
36. $\tan\left(\cos^{-1}\left(\frac{x}{\sqrt{x^2+4}}\right)\right)$

Unit 3 Clusters 2 and 3 HONORS: Solving Trigonometric Equations

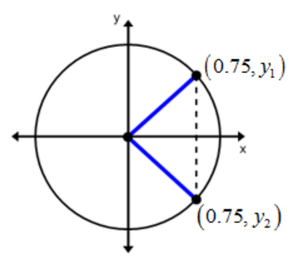
Cluster 2: Extending the domain of trigonometric functions using the unit circle
3.3H Find all solutions for equations involving trigonometric functions.

Example 1:
Find all values of x, $0 \le x < 2\pi$, that make the statement true.

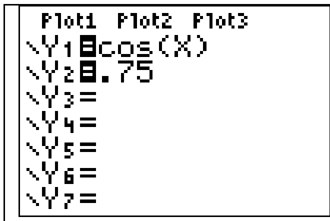
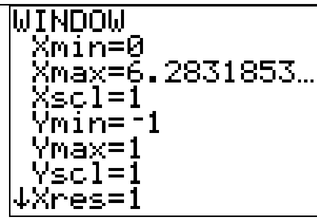
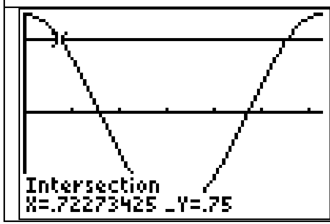
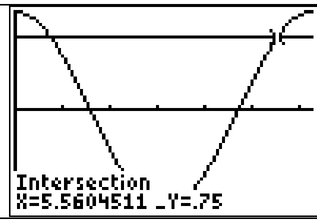
a. $\sin x = \frac{1}{2}$ b. $\cos x = -1$ c. $\csc x = \frac{2}{\sqrt{3}}$

a.	The points that have been rotated $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ radians from the positive *x*-axis have coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ respectively. The *y*-coordinate is the sine value and both points have a *y*-coordinate of $\frac{1}{2}$.
b.	The point that has been rotated π radians from the positive *x*-axis has coordinates $(-1, 0)$. The *x*-coordinate is the cosine value which is -1.
c.	Cosecant is the reciprocal of sine, therefore find all the values that satisfy $\sin x = \frac{\sqrt{3}}{2}$. The angles rotated $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ radians from the positive *x*-axis have coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ respectively. Both have *y*-coordinates of $\frac{\sqrt{3}}{2}$. Both will have a cosecant of $\frac{2}{\sqrt{3}}$.

Example 2:
Use technology to find all solutions in the interval $[0, 2\pi)$ to the trigonometric equation $\cos x = 0.75$.

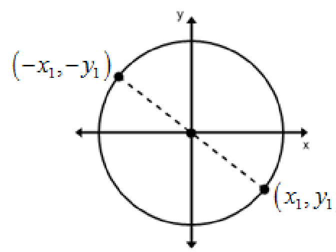
$\cos x = 0.75$	
$x = \cos^{-1}(0.75)$ $x \approx 0.7227$	Make sure that your calculator is in radian mode. Then use the inverse cosine function to find the first value.
$x \approx 2\pi - 0.7227 \approx 5.5605$	The calculator only returns values between 0 and π for inverse cosine. Cosine is positive in the first and fourth quadrants. To get the second value, subtract the first value from 2π. 

This could also be solved graphically.

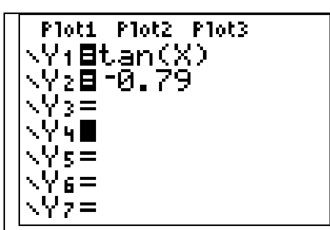
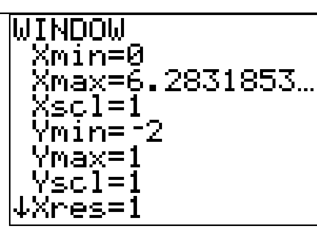
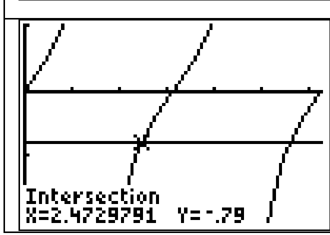
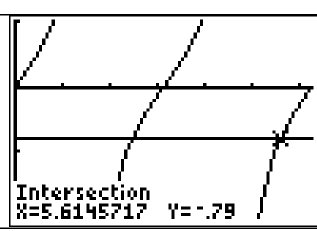
		Enter $y=\cos x$ and $y=0.75$ in your graphing calculator. Set your window so that xmin = 0, xmax = 2π, ymin= -1, and ymax = 1.
		Graph the functions and find both of the intersections.

Example 3:

Use technology to find all solutions in the interval $[0,2\pi)$ to the trigonometric equation $\tan x=-0.79$.

$\tan x=-0.79$	
$x=\tan^{-1}(-0.79)$ $x\approx -0.6686$	Make sure that your calculator is in radian mode. Then use the inverse tangent function to find the first value.
$x\approx -0.6686+\pi\approx 2.4730$	This is not a value between $[0,2\pi)$. The calculator only returns values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for inverse tangent. You need to add π to your answer. To get the value in the second quadrant. 
$x\approx -0.6686+2\pi\approx 5.6146$	Tangent is negative in the second and fourth quadrants. To get the value in the fourth quadrant, add 2π to the value the calculator gave you.

This could also be solved graphically.

		Enter $y=\tan x$ and $y=-0.79$ in your graphing calculator. Set your window so that xmin = 0, xmax = 2π, ymin= -2, and ymax = 1.
		Graph the functions and find both of the intersections.

Practice Exercises A

Find all values of x, $0 \le x < 2\pi$, that make the statement true.

1. $\sin x = -\frac{\sqrt{2}}{2}$	2. $\cos x = \frac{1}{2}$	3. $\tan x = -1$
4. $\sin x = -1$	5. $\cos x = -\frac{\sqrt{3}}{2}$	6. $\tan x = 0$
7. $\sin x = -\frac{\sqrt{3}}{2}$	8. $\cos x = \frac{\sqrt{2}}{2}$	9. $\tan x = \frac{\sqrt{3}}{3}$
10. $\sin x = 0$	11. $\cos x = -1$	12. $\tan x = 1$
13. $\sec x = -\frac{2}{\sqrt{3}}$	14. $\csc x = 2$	15. $\cot x = -1$
16. $\sec x = 1$	17. $\csc x = -1$	18. $\cot x = -\sqrt{3}$

Use technology to find all solutions in the interval $[0, 2\pi)$.

19. $\sin x = 0.33$	20. $\cos x = 0.59$	21. $\tan x = 1.615$
22. $\cos x = -0.36$	23. $\tan x = 0.43$	24. $\sin x = 0.23$
25. $\tan x = -2.4$	26. $\sin x = -0.88$	27. $\cos x = 0.17$

Example 4:
Find all solutions in the interval $[0, 2\pi)$ to the trigonometric equation $2\sin x - 1 = 0$.

$2\sin x - 1 = 0$ $2\sin x = 1$ $\sin x = \frac{1}{2}$	Isolate the trigonometric function.
$x = \sin^{-1}\left(\frac{1}{2}\right)$	Find all angles between 0 and 2π that have a sine value of $\frac{1}{2}$.
$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$	

Example 5:
Find all solutions in the interval $\left[0, 2\pi\right)$ to the trigonometric equation $\sin x\cos x - 3\cos x = 0$.

$\sin x\cos x - 3\cos x = 0$ $\cos x(\sin x - 3) = 0$		Factor out the common term $\cos x$.
$\cos x = 0$	$\sin x - 3 = 0$ $\sin x = 3$	Set each factor equal to zero.
$x = \cos^{-1}\left(0\right)$ $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$	$\sin x \neq 3$	Find all angles between 0 and 2π that have a cosine value of 0. Note: $\sin x$ oscillates between -1 and 1 and will never equal 3.

Example 6:
Find all solutions in the interval $\left[0, 2\pi\right]$ to the trigonometric equation $2\cos^2 x - \cos x - 1 = 0$.

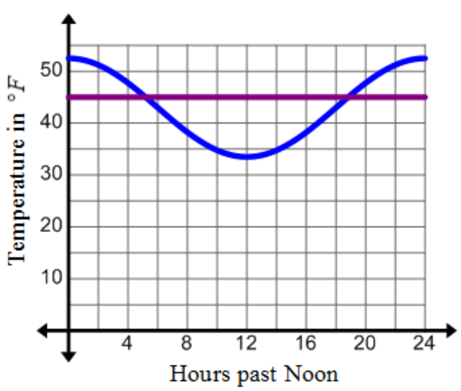
$2\cos^2 x - \cos x - 1 = 0$ $2u^2 - u - 1 = 0$		Let $u = \cos x$ and rewrite the equation.
$\left(2u+1\right)\left(u-1\right) = 0$		Factor the quadratic.
$2u + 1 = 0$ $2u = -1$ $u = -\frac{1}{2}$	$u - 1 = 0$ $u = 1$	Set each factor equal to zero.
$\cos x = -\frac{1}{2}$	$\cos x = 1$	Substitute $\cos x$ back in for *u*.
$x = \cos^{-1}\left(-\frac{1}{2}\right)$ $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$	$x = \cos^{-1}\left(1\right)$ $x = 0$ or $x = 2\pi$	Find all angles between 0 and 2π that have a cosine value of $-\frac{1}{2}$. Find all angles between 0 and 2π inclusive that have a cosine value of 1.

Example 7:
Find all solutions in the interval $\left[0, 2\pi\right)$ to the trigonometric equation $4\cos^2 x\sin x = 3\sin x$.

$4\cos^2 x\sin x = 3\sin x$ $4\cos^2 x\sin x - 3\sin x = 0$ $\sin x\left(4\cos^2 x - 3\right) = 0$	Get all the terms on the same side, then factor out the common term $\sin x$.

$\sin x = 0$	$4\cos^2 x - 3 = 0$ $4\cos^2 x = 3$ $\cos^2 x = \frac{3}{4}$ $\cos x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$	Set each factor equal to zero and isolate the trigonometric expression.
$x = \sin^{-1}(0)$ $x = 0$ or $x = \pi$	$x = \cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$ $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6},$ or $\frac{11\pi}{6}$	Find all angles between 0 and 2π that have a sine value of 0. Find all angles between 0 and 2π that have a cosine value of $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$.

Example 8:
A heater turns on in a home when the outside temperature is below $45°F$. During the middle of March in Salt Lake City, you can model the outside temperature in degrees Fahrenheit using the function $f(t) = 43 + 9.5\cos\frac{\pi}{12}t$, where t is the number of hours past noon. During which hours is the heater heating the home?

	Find where $f(t) = 43 + 9.5\cos\frac{\pi}{12}t$ is equal to $45°F$. The outside temperature is equal to $45°F$ when $t \approx 5.2$ hours past noon and $t \approx 18.8$ hours past noon. 5.2 hours is 5 hours and 12 minutes. $0.2(60) = 12$ 18.8 hours is 18 hours and 48 minutes. $0.8(60) = 48$ The heater is on from 5:12 p.m. to 6:48 a.m.

Practice Exercises B

Find all solutions in the interval $[0, 2\pi)$ to the trigonometric equation.

1. $2\sin x + \sqrt{3} = 0$
2. $3\tan x - \sqrt{3} = 0$
3. $\cos x + \sqrt{2} = -\cos x$
4. $5 + 2\sin x - 7 = 0$
5. $3\cos x = \cos x - 1$
6. $4\sin x = 2\sin x + 1$
7. $4\tan x = 3 + \tan x$
8. $\sqrt{3}\cos x \tan x - \cos x = 0$
9. $2\sin^3 x = \sin x$
10. $\sin x = -\sin x \cos x$
11. $\tan x = \tan^2 x$
12. $2\cos x \sin x - \cos x = 0$
13. $2\sin^2 x - 1 = 0$
14. $4\cos^2 x - 1 = 0$
15. $5\tan^2 x - 15 = 0$
16. $4\cos^2 x - 3 = 0$
17. $\sqrt{2}\tan x \cos x - \tan x = 0$
18. $\sin^2 x - 2\sin x = 0$
19. $\tan x \sin^2 x = \tan x$
20. $3\tan^2 x - 1 = 0$
21. $2\sin^2 x + 3\sin x + 1 = 0$
22. $4\cos^2 x - 4\cos x + 1 = 0$
23. $\sin^2 x + 3\sin x = 0$
24. $2\sin^2 x - 3\sin x = 2$
25. The function $I = 40\sin(60\pi t)$ models the current, *I*, in amps that an electric generator is producing after *t* seconds. How many seconds is the current above 25 amps?
26. The function $h = 25\sin\left(\frac{\pi}{20}(t-10)\right) + 34$ models the height, *h*, of a Ferris wheel car in feet, *t* seconds after starting. On the first rotation only, when is the Ferris wheel 50 feet above ground?
27. The water level of a harbor can be modeled by the equation $f(t) = -30\cos\left(\frac{6\pi}{37}t\right)$, where *t* represents the hours after low tide and *f* is the water depth in feet. Determine how many hours after low tide the water level is at 15 feet during the day.
28. The intensity of a sound wave for a certain pitch fork can be modeled by the function $f(t) = 0.001\sin(1320\pi t)$, where *t* is measured in seconds. When does the intensity first reach -0.0006?

Polar and Parametric

Honors Unit: Polar Coordinates and Equations

Honors Cluster: Polar and Parametric

H1: Define and use polar coordinates and relate them to Cartesian coordinates.

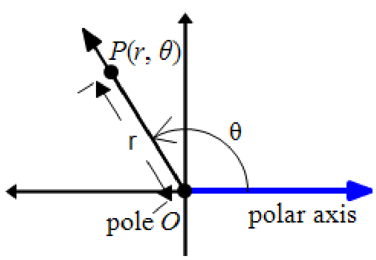
H3: Translate equations in Cartesian coordinates into polar coordinates and graph equations in the polar coordinate plane.

Polar Coordinates

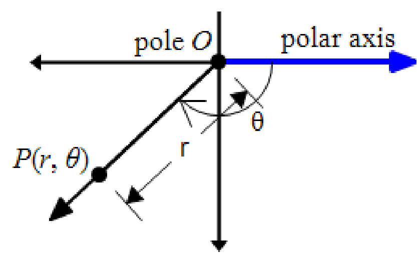
VOCABULARY

In addition to the Cartesian plane and the complex plane, there is another coordinate plane called the **polar coordinate system**. The origin of the polar coordinate system, *O*, is called the **pole**. The pole is the endpoint of a ray, called the **polar axis**, which extends to the right. A point, *P*, on the polar coordinate system is represented by (r,θ), where *r* is the directed distance from the pole, *O*, to point *P* and θ is the angle of rotation with initial side the polar axis and terminal side the segment OP. Just like in trigonometry, positive angles rotate counterclockwise from the polar axis and negative angles rotate clockwise from the polar axis.

Positive Rotation



Negative Rotation

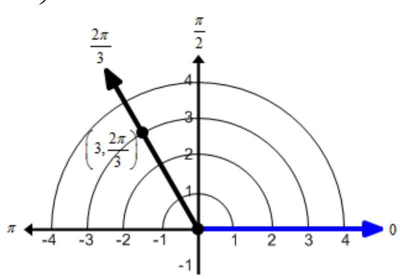


Example 1:

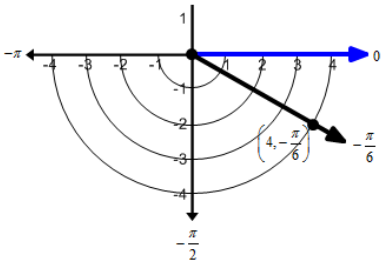
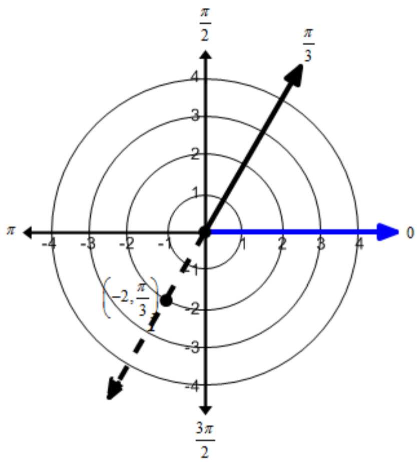
Plot the following points on the polar coordinate system.

a. $\left(3,\dfrac{2\pi}{3}\right)$ b. $\left(4,-\dfrac{\pi}{6}\right)$ c. $\left(-2,\dfrac{\pi}{3}\right)$

a. $\left(3,\dfrac{2\pi}{3}\right)$



Rotate the ray on the polar axis $\dfrac{2\pi}{3}$ radians counterclockwise and go out 3 units from the pole.

b. $\left(4,-\frac{\pi}{6}\right)$ 	Rotate the ray on the polar axis $\frac{\pi}{6}$ radians clockwise and go out 4 units from the pole.
c. $\left(-2,\frac{\pi}{3}\right)$ 	Rotate the ray on the polar axis $\frac{\pi}{3}$ radians counterclockwise and go out 2 units on the ray opposite the terminal side of the angle.

Unlike rectangular coordinates, (x,y), that have exactly one representation, polar coordinates can be represented in infinitely many ways. Take a closer look at Example 1 part c. The point $\left(-2,\frac{\pi}{3}\right)$ could also have been written as $\left(2,\frac{4\pi}{3}\right)$ or even $\left(-2,\frac{7\pi}{3}\right)$. To get the point $\left(2,\frac{4\pi}{3}\right)$ from the point $\left(-2,\frac{\pi}{3}\right)$, r was changed to a positive number and π was added to θ. To get the point $\left(-2,\frac{7\pi}{3}\right)$ from the point $\left(-2,\frac{\pi}{3}\right)$, 2π was added to θ. The table below summarizes how multiple representations of the same polar point can be found.

Finding Multiple Representations of Points
The point (r,θ) can be represented as $(r,\theta+2n\pi)$ or $(-r,\theta+\pi+2n\pi)=(-r,\theta+(2n+1)\pi)$, where n is an integer.

Example 2:
For each polar coordinate, find another representation in which:

1. r is positive and θ is on the interval $(2\pi, 4\pi)$.
2. r is negative and θ is on the interval $(0, 2\pi)$.
3. r is positive and θ is on the interval $(-2\pi, 0)$.

a. $\left(3, \frac{\pi}{4}\right)$

b. $\left(5, \frac{2\pi}{3}\right)$

a.

1. $\left(3, \frac{\pi}{4}\right) = \left(3, \frac{\pi}{4} + 2\pi\right) = \left(3, \frac{9\pi}{4}\right)$ Add 4π to θ.
2. $\left(3, \frac{\pi}{4}\right) = \left(-3, \frac{\pi}{4} + \pi\right) = \left(-3, \frac{5\pi}{4}\right)$ Change the sign of r and add π to θ.
3. $\left(3, \frac{\pi}{4}\right) = \left(3, \frac{\pi}{4} - 2\pi\right) = \left(3, -\frac{7\pi}{4}\right)$ Subtract 2π from θ.

b.

1. $\left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} + 2\pi\right) = \left(5, \frac{8\pi}{3}\right)$ Add 2π to θ.
2. $\left(5, \frac{2\pi}{3}\right) = \left(-5, \frac{2\pi}{3} + \pi\right) = \left(-5, \frac{5\pi}{3}\right)$ Change the sign of r and add π to θ.
3. $\left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} - 2\pi\right) = \left(5, -\frac{4\pi}{3}\right)$ Subtract 2π from θ.

Relating Polar and Cartesian Coordinates

A polar point $P(r, \theta)$ is equal to a rectangular point $P(x, y)$ when the pole is the origin and the polar axis is the positive x-axis. Using trigonometry, equations can be derived that relate the polar coordinates to the rectangular coordinates of a point P.

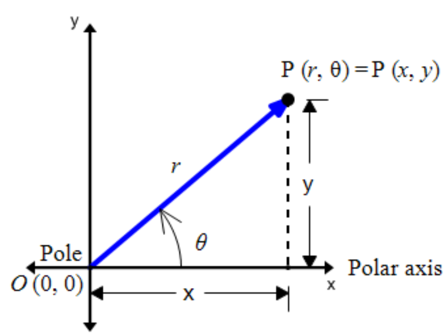
Relating Polar and Cartesian (Rectangular) Coordinates

$\cos\theta = \frac{x}{r}$ therefore $x = r\cos\theta$

$\sin\theta = \frac{y}{r}$ therefore $y = r\sin\theta$

$\tan\theta = \frac{y}{x}$ therefore $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r^2 = x^2 + y^2$ by the Pythagorean Theorem.



Example 3:
Find the rectangular coordinates of the given polar coordinates.

a. $\left(2, \frac{5\pi}{6}\right)$ b. $\left(-4, \frac{4\pi}{3}\right)$

a. $\left(2, \frac{5\pi}{6}\right)$	$r = 2$ and $\theta = \frac{5\pi}{6}$
$x = r\cos\theta$ $\quad y = r\sin\theta$ $x = 2\cos\frac{5\pi}{6}$ $\quad y = 2\sin\frac{5\pi}{6}$ $x = 2\left(-\frac{\sqrt{3}}{2}\right)$ $\quad y = 2\left(\frac{1}{2}\right)$ $x = -\sqrt{3}$ $\quad y = 1$	Use $x = r\cos\theta$ and $y = r\sin\theta$ to find the rectangular coordinates.
$\left(2, \frac{5\pi}{6}\right)$ in rectangular coordinates is $\left(-\sqrt{3}, 1\right)$.	

b. $\left(-4, \frac{4\pi}{3}\right)$	$r = -4$ and $\theta = \frac{4\pi}{3}$
$x = r\cos\theta$ $\quad y = r\sin\theta$ $x = -4\cos\frac{4\pi}{3}$ $\quad y = -4\sin\frac{4\pi}{3}$ $x = -4\left(-\frac{\sqrt{2}}{2}\right)$ $\quad y = -4\left(-\frac{\sqrt{2}}{2}\right)$ $x = 2\sqrt{2}$ $\quad y = 2\sqrt{2}$	Use $x = r\cos\theta$ and $y = r\sin\theta$ to find the rectangular coordinates.
$\left(-4, \frac{4\pi}{3}\right)$ in rectangular coordinates is $\left(2\sqrt{2}, 2\sqrt{2}\right)$.	

Example 4:
Find the polar coordinates of the given rectangular coordinates.

a. $(3, -3)$ b. $\left(-6, -2\sqrt{3}\right)$

a. $(3, -3)$ $r^2 = (3)^2 + (-3)^2$ $r^2 = 9 + 9$ $r^2 = 18$ $r = \pm\sqrt{18}$ $r = 3\sqrt{2}$	Use $r^2 = x^2 + y^2$ to find the radius. Use the positive value for *r*.

$\theta = \tan^{-1}\left(\frac{-3}{3}\right)$ $\theta = \tan^{-1}(-1)$ $\theta = \frac{7\pi}{4}$	Use $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ to find θ. Find the positive angle that is in the fourth quadrant (the same quadrant as the rectangular point).
$(3,-3)$ in polar coordinates is $\left(3\sqrt{2}, \frac{7\pi}{4}\right)$.	

b. $\left(-6,-2\sqrt{3}\right)$ $r^2 = (-6)^2 + \left(-2\sqrt{3}\right)^2$ $r^2 = 36+12$ $r^2 = 48$ $r = \pm\sqrt{48}$ $r = 4\sqrt{3}$	Use $r^2 = x^2 + y^2$ to find the radius. Use the positive value for *r*.
$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-6}\right)$ $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ $\theta = \frac{7\pi}{6}$	Use $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ to find θ. Find the positive angle that is in the third quadrant (the same quadrant as the rectangular point).
$\left(-6,-2\sqrt{3}\right)$ in polar coordinates is $\left(4\sqrt{3}, \frac{7\pi}{6}\right)$.	

Practice Exercises A

Plot the following points on the polar coordinate system.

1. $\left(3, \frac{4\pi}{3}\right)$
2. $\left(2, \frac{5\pi}{6}\right)$
3. $\left(-1, \frac{3\pi}{4}\right)$
4. $(-4, \pi)$
5. $\left(5, -\frac{7\pi}{6}\right)$
6. $\left(-2, -\frac{\pi}{3}\right)$
7. $(-1, 0)$
8. $\left(\frac{3}{2}, \frac{3\pi}{2}\right)$

For each polar coordinate, find another representation in which:

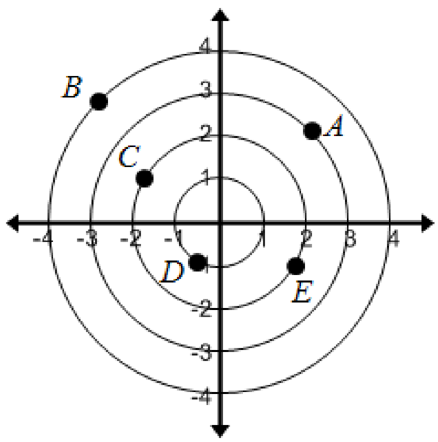
a. r is positive and θ is on the interval $[2\pi, 4\pi)$.

b. r is negative and θ is on the interval $[0, 2\pi)$.

c. r is positive and θ is on the interval $[-2\pi, 0)$.

9. $\left(5, \frac{\pi}{6}\right)$
10. $(7, \pi)$
11. $\left(10, \frac{3\pi}{4}\right)$
12. $\left(2, \frac{4\pi}{3}\right)$
13. $\left(12, \frac{5\pi}{4}\right)$
14. $\left(8, \frac{5\pi}{6}\right)$
15. $\left(3, \frac{\pi}{2}\right)$
16. $\left(4, \frac{5\pi}{3}\right)$

Identify the letter that represents the given polar coordinate.



17. $\left(-1, \frac{\pi}{3}\right)$
18. $\left(-4, -\frac{\pi}{4}\right)$
19. $\left(3, -\frac{7\pi}{4}\right)$
20. $\left(2, -\frac{\pi}{6}\right)$
21. $\left(4, -\frac{5\pi}{4}\right)$
22. $\left(1, \frac{10\pi}{3}\right)$
23. $\left(-2, \frac{11\pi}{6}\right)$
24. $\left(-3, -\frac{3\pi}{4}\right)$

Find the rectangular coordinates of the given polar coordinates.

25. $\left(4, \frac{\pi}{2}\right)$
26. $(6, \pi)$
27. $\left(2, \frac{\pi}{3}\right)$
28. $\left(4, \frac{5\pi}{6}\right)$
29. $\left(-6, \frac{3\pi}{4}\right)$
30. $\left(3, \frac{3\pi}{2}\right)$
31. $\left(5, \frac{7\pi}{6}\right)$
32. $(-3, 0)$

Find the polar coordinates of the given rectangular coordinates.

33. $(-2, 2)$
34. $(2, -2)$
35. $\left(2, -2\sqrt{3}\right)$
36. $\left(-2\sqrt{3}, 2\right)$
37. $\left(-\sqrt{3}, -1\right)$
38. $\left(-1, -\sqrt{3}\right)$
39. $(-3, 0)$
40. $(0, 4)$

Translating Equations in Cartesian Coordinates to Polar Coordinates

The variables of a polar equation are r and θ. When converting a Cartesian equation to a polar equation use the conversion equations: $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$.

Example 5:
Convert the following Cartesian equations to polar equations.

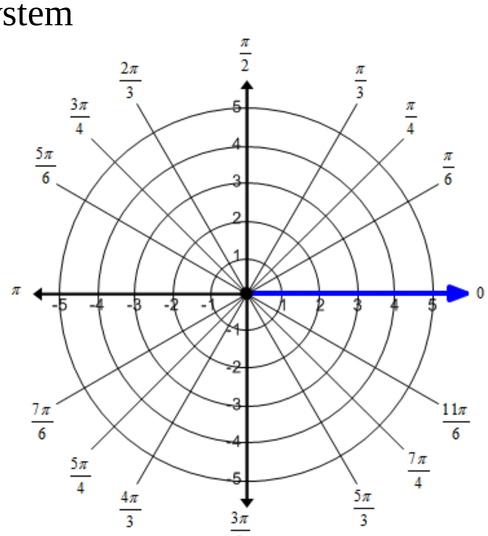
a. $x = 5$ b. $-3x + 4y = 6$ c. $x^2 + (y-4)^2 = 16$

a. $x = 5$ $r\cos\theta = 5$ $r = \dfrac{5}{\cos\theta}$	Substitute $x = r\cos\theta$ then solve the equation for *r*.
b. $-3x + 4y = 6$ $-3r\cos\theta + 4r\sin\theta = 6$ $r(-3\cos\theta + 4\sin\theta) = 6$ $r = \dfrac{6}{-3\cos\theta + 4\sin\theta}$	Substitute $x = r\cos\theta$ and $y = r\sin\theta$ then solve the equation for *r*.
c. $x^2 + (y-4)^2 = 16$ $x^2 + y^2 - 8y + 16 = 16$ $x^2 + y^2 - 8y = 0$ $r^2 - 8r\sin\theta = 0$ $r(r - 8\sin\theta) = 0$ $r = 0$ $r - 8\sin\theta = 0$ $r = 8\sin\theta$	Expand the equation and simplify. Substitute $x^2 + y^2 = r^2$ and $y = r\sin\theta$ then solve the equation for *r*. Set each factor equal to zero. $r = 0$ is a point at the pole. The equation $r = 8\sin\theta$ passes through this point so $r = 8\sin\theta$ is the only equation needed.

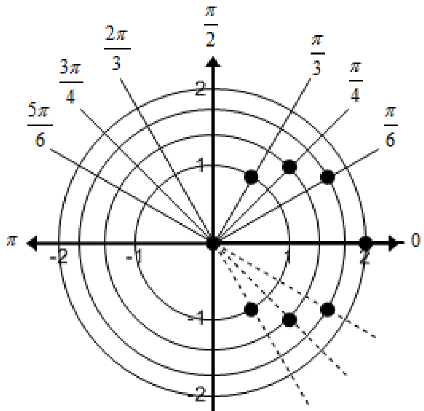
Graphing Equations on the Polar Coordinate System

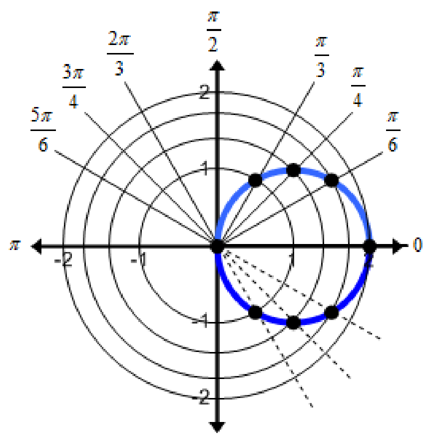
Graphing on the polar coordinate system is like graphing on a set of concentric circles whose center is the pole. The independent variable is θ and the dependent variable is *r*. The graph is made by plotting all the points that are solutions to the equation.

The graph at the right shows lines going through the pole at angle values for which exact trigonometric values are known. Using these angles makes it easier to graph the polar equations by hand.



Example 6:
Graph the equation $r = 2\cos\theta$.





Evaluate the equation at each value of θ then plot the points on the polar coordinate system.

θ	$r = 2\cos\theta$
0	2
$\frac{\pi}{6}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	-1
$\frac{3\pi}{4}$	$-\sqrt{2}$
$\frac{5\pi}{6}$	$-\sqrt{3}$
π	-2

Connect the points in the order that they were plotted.

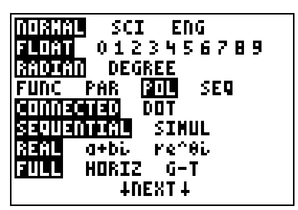
Example 7:
Use technology to graph the equation $r = 4\sin(3\theta)$.

First you will need to change the mode on your calculator so that you are in polar mode (make sure that radians is selected). Push 

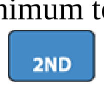
then arrow down to FUNC PAR POL SEQ. Use your arrow keys to arrow over to POL and push 

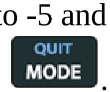
so that polar is selected. Exit the screen by pushing 

.



Next you will need to set the window for your equation. Push 

. There are a few new items when graphing in polar mode. Typically θ will need to have a minimum of 0 and a maximum of 2π. Set the x minimum to -7 and the x maximum to 7. Set the y minimum to -5 and the y maximum to 5. Exit the screen by pushing 



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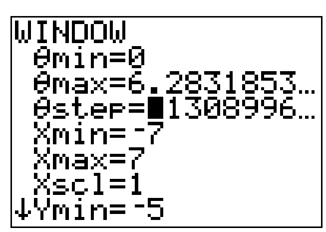
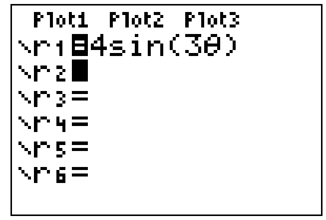


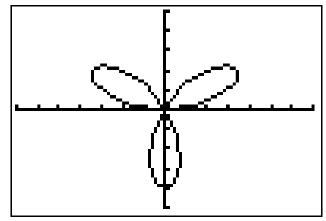
Enter the equation by pushing 

. The variable button 

will now be θ. Once you have entered the equation, push 

and you should get something that looks like the graph at the right.





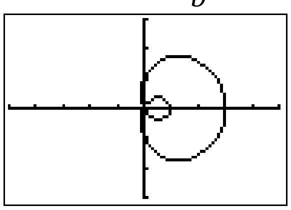
Types of Polar Graphs

Limacons

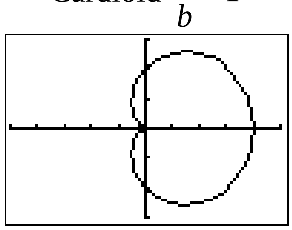
Equations: $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$ where $a > 0$, $b > 0$

Graphs:

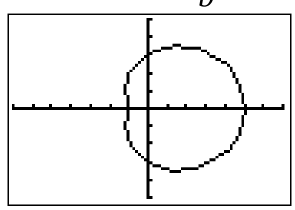
Inner Loop $\frac{a}{b} < 1$



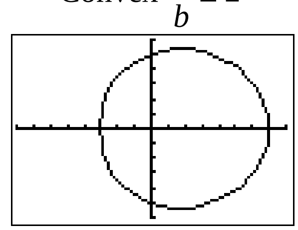
Cardioid $\frac{a}{b} = 1$



Dimpled $1 < \frac{a}{b} < 2$



Convex $\frac{a}{b} \geq 2$

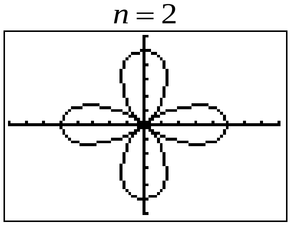


Rose Curves

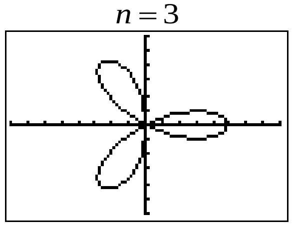
Equations: $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ where $n > 1$ is an integer and $a \neq 0$. The number of petals is determined by the value of *n*. If *n* is even then there will be 2*n* petals. If *n* is odd then there will be *n* petals.

Graphs:

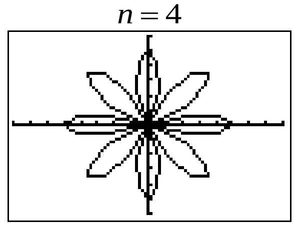
$n = 2$



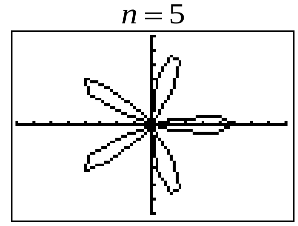
$n = 3$



$n = 4$



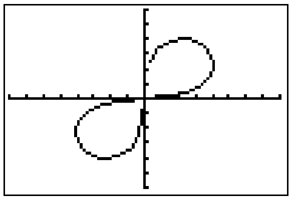
$n = 5$

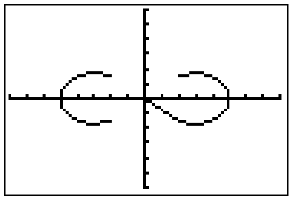


Lemniscates

Equations: $r^2 = a^2 \sin 2\theta$ or $r^2 = a^2 \cos 2\theta$, $a \neq 0$

Graphs:





Practice Exercises B

Convert each Cartesian equation to a polar equation.

1. $x = -3$
2. $x = 5$
3. $y = 4$
4. $y = -8$
5. $2x - 3y = 5$
6. $3x + 4y = 2$
7. $x + 5y = 8$
8. $xy = 4$
9. $x^2 + y^2 = 25$
10. $x^2 + y^2 = 4y$
11. $x^2 + y^2 = 6x$
12. $y^2 = 8x$
13. $x^2 = 3y$
14. $(y+1)^2 = 2x + 1$
15. $(x+4)^2 = 4y + 16$
16. $(x-2)^2 + y^2 = 4$
17. $(x+3)^2 + (y+3)^2 = 18$
18. $(x-1)^2 + (y+4)^2 = 17$

Use technology to graph the following equations on the polar coordinate system.

19. $r = 3\sin\theta$
20. $r = 4\cos\theta$
21. $r = 1 - \sin\theta$
22. $r = 1 + \sin\theta$
23. $r = 2 + \cos\theta$
24. $r = 2 - \sin\theta$
25. $r = 2 - 3\sin\theta$
26. $r = 4\sin 2\theta$
27. $r = 5\cos 3\theta$
28. $r = 2\sin 4\theta$
29. $r^2 = 9\cos 2\theta$
30. $r^2 = 9\sin 2\theta$

Honors Unit: Polar Coordinates and Complex Numbers

H2: Represent complex numbers in rectangular and polar form and convert between rectangular and polar form.
H4: Multiply complex numbers in polar form and use DeMoivre's Theorem to find roots of complex numbers.

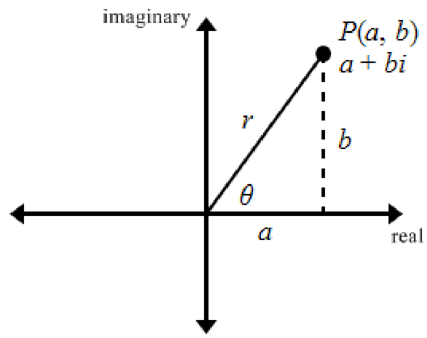
Polar Form of a Complex Number

VOCABULARY

The **absolute value** of a complex number, $a+bi$, is $|z|=\sqrt{a^2+b^2}$.

The **polar form** (sometimes called trigonometric form) of a complex number is represented by $z=r(\cos\theta+i\sin\theta)$ where $a=r\cos\theta$, $b=r\sin\theta$, r is the modulus (absolute value) of a complex number, and θ is an argument of *z*.

An **argument** of a complex number is the direction angle of the vector $\langle a,b\rangle$. The angle, θ, can be found using $\theta=\tan^{-1}\left(\frac{b}{a}\right)$.



Example 1:
Convert the complex numbers in rectangular form to polar form with θ on the interval $[0,2\pi)$.

a. $z=-1+\sqrt{3}i$ b. $z=3-4i$

a. $z=-1+\sqrt{3}i$ $r=\lvert z\rvert=\sqrt{(-1)^2+(\sqrt{3})^2}$ $r=\sqrt{1+3}$ $r=\sqrt{4}$ $r=2$	Find the modulus or absolute value. $a=-1$ $b=\sqrt{3}$
$\theta=\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ $\theta=\tan^{-1}(-\sqrt{3})$ $\theta=\frac{2\pi}{3}$	Find θ using $\theta=\tan^{-1}\left(\frac{b}{a}\right)$. The angle should be in the second quadrant.
$z=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$	Use $z=r(\cos\theta+i\sin\theta)$ and substitute in known values.

b. $z=3-4i$ $r=\|z\|=\sqrt{(3)^2+(-4)^2}$ $r=\sqrt{9+16}$ $r=\sqrt{25}$ $r=5$	Find the modulus or absolute value. $a=3\ \ b=-4$
$\theta=\tan^{-1}\left(\dfrac{-4}{3}\right)$ $\theta\approx-0.927$ $\theta\approx-0.927+2\pi$ $\theta\approx5.356$	Find θ using $\theta=\tan^{-1}\left(\dfrac{b}{a}\right)$. The angle should be in the fourth quadrant. Add 2π to the answer your calculator gives you.
$z=5(\cos 5.356+i\sin 5.356)$	Use $z=r(\cos\theta+i\sin\theta)$ and substitute in known values.

Example 2:
Write each complex number in rectangular form.

a. $z=4\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$ b. $z=3\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)$

a. $z=4\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$ $z=4\left(-\dfrac{\sqrt{3}}{2}+i\dfrac{1}{2}\right)$ $z=-2\sqrt{3}+2i$	Evaluate cosine and sine at the angle. Distribute the modulus.

b. $z=3\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)$ $z=3(0+i(1))$ $z=0+3i$	Evaluate cosine and sine at the angle. Distribute the modulus.

Multiplying Complex Numbers in Polar Form

The Product of Two Complex Numbers in Polar Form

To multiply two complex numbers, multiply the moduli and add the arguments. Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ then the product is $z_1\cdot z_2=r_1\cdot r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$.

Example 3:
Multiply the complex numbers in polar form, then write the result in standard form.

a. $z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_2 = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

b. $z_1 = 5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z_2 = 4\left(\cos\pi + i\sin\pi\right)$

a. $z_1 \cdot z_2 = 2\cdot 3\left[\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)\right]$ $z_1 \cdot z_2 = 6\left[\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right]$	Multiply the moduli and add the arguments.
$z_1 \cdot z_2 = 6\left[-0.966+i\left(0.259\right)\right]$ $z_1 \cdot z_2 \approx -5.796+1.553i$	Evaluate cosine and sine at the angle. Distribute the modulus.

b. $z_1 \cdot z_2 = 5\cdot 4\left[\cos\left(\frac{\pi}{2}+\pi\right)+i\sin\left(\frac{\pi}{2}+\pi\right)\right]$ $z_1 \cdot z_2 = 20\left[\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right]$	Multiply the moduli and add the arguments.
$z_1 \cdot z_2 = 20\left[0+i\left(-1\right)\right]$ $z_1 \cdot z_2 = -20i$	Evaluate cosine and sine at the angle. Distribute the modulus.

Finding Roots of Complex Numbers

De Moivre's Theorem for Complex Roots

Let n be an integer greater than 1 and $z = r\left(\cos\theta + i\sin\theta\right)$ be any complex number. Then z has n distinct nth roots as follows:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, ..., n-1$

Example 4:
Find the 5 complex roots of $1+i$. Write the roots in polar form, with θ in radians.

$r=\sqrt{1^2+1^2}=\sqrt{2}$, $\theta=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4}$ $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$	Change $1+i$ into polar form.
1st Root $k=0$ $z_0=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(0)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(0)\pi}{5}\right)\right]$ $z_0=\sqrt[10]{2}\left(\cos\frac{\pi}{20}+i\sin\frac{\pi}{20}\right)$	$\theta=\frac{\pi}{4}$, $n=5$, and $\sqrt[5]{\sqrt{2}}=\sqrt[10]{2}$. Substitute in known information. Simplify.
2nd Root $k=1$ $z_1=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(1)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(1)\pi}{5}\right)\right]$ $z_1=\sqrt[10]{2}\left(\cos\frac{9\pi}{20}+i\sin\frac{9\pi}{20}\right)$	
3rd Root $k=2$ $z_2=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(2)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(2)\pi}{5}\right)\right]$ $z_2=\sqrt[10]{2}\left(\cos\frac{17\pi}{20}+i\sin\frac{17\pi}{20}\right)$	
4th Root $k=3$ $z_3=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(3)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(3)\pi}{5}\right)\right]$ $z_3=\sqrt[10]{2}\left(\cos\frac{25\pi}{20}+i\sin\frac{25\pi}{20}\right)$ $z_3=\sqrt[10]{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$	
5th Root $k=4$ $z_4=\sqrt[10]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2(4)\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{4}+2(4)\pi}{5}\right)\right]$ $z_4=\sqrt[10]{2}\left(\cos\frac{33\pi}{20}+i\sin\frac{33\pi}{20}\right)$	

Example 5:
Find the complex cube roots of 27. Write the roots in rectangular form.

$r=\sqrt{27^2+0^2}=27$, $\theta=\tan^{-1}\left(\frac{0}{27}\right)=0$ $27=27(\cos 0+i\sin 0)$	Change 27 into polar form.
1st Root $k=0$ $z_0=3\left[\cos\left(\frac{0+2(0)\pi}{3}\right)+i\sin\left(\frac{0+2(0)\pi}{3}\right)\right]$ $z_0=3(\cos 0+i\sin 0)$ $z_0=3(1+i\cdot 0)=3$	$\theta=0$, $n=3$, and $\sqrt[3]{27}=3$. Substitute in known information. Simplify.
2nd Root $k=1$ $z_1=3\left[\cos\left(\frac{0+2(1)\pi}{3}\right)+i\sin\left(\frac{0+2(1)\pi}{3}\right)\right]$ $z_1=3\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ $z_1=3\left(-\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right)=-\frac{3}{2}+i\frac{3\sqrt{3}}{2}$	
3rd Root $k=2$ $z_2=3\left[\cos\left(\frac{0+2(2)\pi}{3}\right)+i\sin\left(\frac{0+2(2)\pi}{3}\right)\right]$ $z_2=3\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$ $z_2=3\left(-\frac{1}{2}+i\cdot-\frac{\sqrt{3}}{2}\right)=-\frac{3}{2}-i\frac{3\sqrt{3}}{2}$	

Practice Exercises A

Convert the complex numbers in rectangular form to polar form with θ on the interval $[0,2\pi)$.

1. $3+3i$
2. $1+\sqrt{3}i$
3. $-2-2i$
4. $3-3i$
5. $-5i$
6. $2\sqrt{3}-2i$
7. $-2+2\sqrt{3}i$
8. -4
9. $-3\sqrt{3}+3i$
10. $-4+3i$
11. $-2-3i$
12. $1-\sqrt{5}i$

Write each complex number in rectangular form.

13. $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

14. $6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

15. $8\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$

16. $10\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

17. $4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

18. $2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

19. $7\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

20. $5\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

21. $3(\cos\pi+i\sin\pi)$

Multiply the complex numbers in polar form, then write the result in standard form. If necessary, round to three decimal places.

22. $z_1=5\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$
$z_2=3\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

23. $z_1=\sqrt{3}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$
$z_2=\frac{1}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

24. $z_1=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$
$z_2=4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

25. $z_1=2\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$
$z_2=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

26. $z_1=3(\cos\pi+i\sin\pi)$
$z_2=8\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

27. $z_1=\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$
$z_2=2\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

28. $z_1=7\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)$
$z_2=2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

29. $z_1=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$
$z_2=\frac{1}{2}\left(\cos\left(-\frac{\pi}{9}\right)+i\sin\left(-\frac{\pi}{9}\right)\right)$

Find the complex roots of each complex number. Write the roots in polar form, with θ in radians.

30. $1-i$, $n=4$

31. $\sqrt{3}-i$, $n=3$

32. $2+2\sqrt{3}i$, $n=5$

33. $-4+4i$, $n=3$

34. $-2\sqrt{3}-2i$, $n=5$

35. $9+3\sqrt{3}i$, $n=4$

Find the complex roots of each number. Write the roots in rectangular form.

36. 64, $n=3$

37. 8, $n=3$

38. 32, $n=5$

Honors Unit: Parametric Equations

H5: Define a curve parametrically and draw parametric graphs.

VOCABULARY

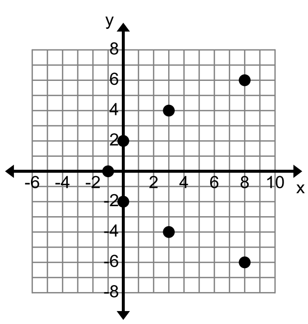
A **parametric curve** is the graph of the ordered pairs (x, y) where $x = f(t)$ and $y = g(t)$. $f(t)$ and $g(t)$ are functions defined on an interval *I*, the **parameter interval**, of *t*-values. The variable *t* is called a **parameter**, and the equations $x = f(t)$ and $y = g(t)$ are **parametric equations** for the curve. The parameter *t* often represents time. The **orientation** of a parametric curve is the direction the curve moves in as *t* increases. Little arrows are drawn on the graph to indicate the orientation.

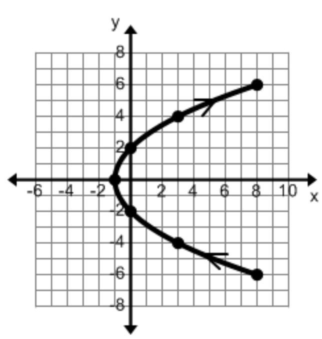
Example 1:
Graph the parametric curve defined by the parametric equations.

$$x = t^2 - 1,\ y = 2t,\ -3 \le t \le 3$$

t	$x = t^2 - 1$	$y = 2t$	(x, y)
-3	$(-3)^2 - 1 = 8$	$2(-3) = -6$	$(8, -6)$
-2	$(-2)^2 - 1 = 3$	$2(-2) = -4$	$(3, -4)$
-1	$(-1)^2 - 1 = 0$	$2(-1) = -2$	$(0, -2)$
0	$(0)^2 - 1 = -1$	$2(0) = 0$	$(-1, 0)$
1	$(1)^2 - 1 = 0$	$2(1) = 2$	$(0, 2)$
2	$(2)^2 - 1 = 3$	$2(2) = 4$	$(3, 4)$
3	$(3)^2 - 1 = 8$	$2(3) = 6$	$(8, 6)$

Construct a table of values.



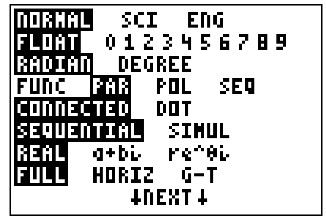
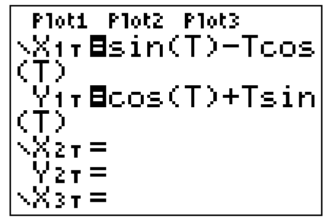
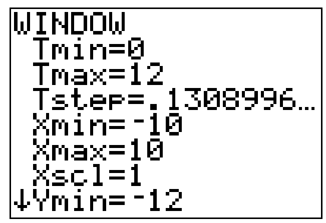
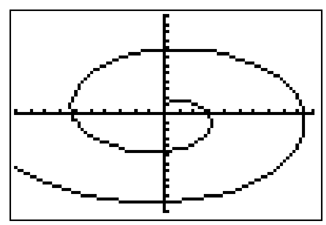


Plot the points.

Connect the points with a smooth curve indicating the orientation of the curve.

Example 2:
Use technology to graph the curve defined parametrically.

$$x = \sin t - t\cos t,\ \ y = \cos t + t\sin t,\ \ 0 \le t \le 12$$

<table>
<tr><td>You will need to be in parametric mode. Push MODE and use your arrow keys to arrow down to FUNC PAR POL SEQ. Then arrow over to PAR and push ENTER to select it. The calculator will now be in parametric mode. Exit this screen by pushing 2ND QUIT.</td><td>
</td></tr>
<tr><td>Enter the equations by pushing Y=. The variable button X,T,θ,n will now be T. You will need to enter the x equation in X1 and the y equation in Y1. Both equations must be entered before the parametric curve can be graphed.</td><td>
</td></tr>
<tr><td>Next you will need to set your window. Push WINDOW. The parameter T should have a minimum of 0 and a maximum of 12. You will have to determine the window that you like for the x and the y, but xmin at -10 and xmax at 10 and ymin at -12 and ymax at 12 will give you a nice view of the curve.</td><td>
</td></tr>
<tr><td>Graph the equation by pushing GRAPH. Notice the orientation of the curve as it is graphed.</td><td></td></tr>
</table>

The graph in Example 1 is a parabola that opens to the right. This is a curve that we could have graphed without being in parametric mode. It is sometimes possible to eliminate the parameter and write an equivalent equation that is in terms of *x* and *y*. However, you may need to change the domain of the rectangular equation so that it is consistent with the domain of the parametric equation.

Example 3:
Eliminate the parameter of the curve and identify the graph of the parametric curve.

$$x = \sqrt{t},\ \ y = 2t + 1,\ \ 0 < t < 9$$

$x = \sqrt{t}$ $x^2 = t$	Solve the *x* equation for *t*.

$y = 2t + 1$ $y = 2\left(x^2\right) + 1$ $y = 2x^2 + 1$	Substitute the value of *t* in to the *y* equation.
The graph is the right side of a parabola that opens up. It has domain $[0,3]$.	

Example 4:
Eliminate the parameter of the curve and identify the graph of the parametric curve.

$$x = 3\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$$

$x = 3\cos t$ $\dfrac{x}{3} = \cos t$	$y = 2\sin t$ $\dfrac{y}{2} = \sin t$	Use the identity $\sin^2 t + \cos^2 t = 1$ to eliminate the parameter. First, isolate the trigonometric term in each equation.
$\left(\dfrac{x}{3}\right)^2 = \cos^2 t$ $\dfrac{x^2}{9} = \cos^2 t$	$\left(\dfrac{y}{2}\right)^2 = \sin^2 t$ $\dfrac{y^2}{4} = \sin^2 t$	Square both sides of each equation.
$\dfrac{x^2}{9} = \cos^2 t$ $+\dfrac{y^2}{4} = \sin^2 t$ $\dfrac{x^2}{9} + \dfrac{y^2}{4} = \cos^2 t + \sin^2 t$ $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$		Add the two new equations together and simplify.
The equation is an ellipse centered at $(0,0)$. The domain is $[-3,3]$.		

Finding Parametric Equations

Finding Parametric Equations for a Function Given a function $y = f(x)$, the parametric equations $x = t$ and $y = f(t)$, where *t* is in the domain of *f*, can be used to define the plane curve. It is important to know that there are infinitely many pairs of parametric equations that can represent a single plane curve. The *x* equation can equal anything that will allow it to take on all the values in the domain of the original function.

Example 5:
Find three different sets of parametric equations for the function whose equation is $f(x)=x^2-3$.

Set 1: $x=t$ $y=t^2-3$	Let x equal t.
Set 2: $x=t-1$ $y=(t-1)^2-3$ $y=t^2-2t+1-3$ $y=t^2-2t-2$	Let x equal $t-1$.
Set 3: $x=t^3$ $y=\left(t^3\right)^2-3$ $y=t^6-3$	Let x equal t^3.

Applications of Parametric Equations

> Projectile Motion
>
> The parametric equations $x=(v_0\cos\theta)t$ and $y=(v_0\sin\theta)t+k-16t^2$ represent the horizontal and vertical distances, in feet, of a projectile that has been launched k feet above ground at an angle θ to the horizontal and an initial velocity of v_0 feet per second. The parameter t represents the time in seconds.

Example 6:
A baseball was hit with an initial velocity of 160 feet per second at an angle of $38°$ to the horizontal. The ball was hit at a height of 3 feet above the ground.

a. Find the parametric equations that describe the position of the ball.
b. When does the ball hit the ground?
c. How far from its starting point does it land?
d. What is the maximum height of the ball during its flight?

a. $x=(v_0\cos\theta)t$ $x=(160\cos 38°)t$ $y=(v_0\sin\theta)t+k-16t^2$ $y=(160\sin 38°)t+3-16t^2$	Use $v_0=160$, $\theta=38°$, $k=3$, $x=(v_0\cos\theta)t$, and $y=(v_0\sin\theta)t+k-16t^2$.

b. $0=(160\sin 38°)t+3-16t^2$ $t=\dfrac{-160\sin 38°\pm\sqrt{(160\sin 38°)^2-4(-16)(3)}}{2(-16)}$ $t\approx -0.030$ or 6.187 The ball will hit the ground after 6.187 seconds.	The vertical height is equal to zero when the ball hits the ground. Set the *y* equation equal to zero and solve for *t*. (Make sure that your calculator is in degrees.) Use the positive value of *t*.
c. $x=(160\cos 38°)t$ $x=(160\cos 38°)(6.187)$ $x\approx 780.1$ The ball will land about 780.1 feet from where it was hit.	Evaluate the *x* equation at the time found in part b.
d. $t=-\dfrac{160\sin 38°}{2(-16)}\approx 3.078$ seconds $y=(160\sin 38°)t+3-16t^2$ $y=(160\sin 38°)(3.078)+3-16(3.078)^2$ $y\approx 154.6$ The ball will reach a maximum height of about 154.6 feet.	Use the *y* equation. The maximum height will be reached when $t=-\dfrac{b}{2a}$. Evaluate the *y* equation at this time.

Practice Exercises A

Graph the curve defined parametrically. Be sure to indicate the orientation.

1. $x=t+2,\ y=t^2;\ -2\le t\le 2$
2. $x=t-2\ \ y=2t+1;\ -2\le t\le 3$
3. $x=t+1,\ y=\sqrt{t};\ t\ge 0$
4. $x=\sqrt{t},\ y=t-1;\ t\ge 0$
5. $x=2\cos t,\ y=2\sin t;\ 0\le t<2\pi$
6. $x=-3\cos t,\ y=5\sin t;\ 0\le t<2\pi$
7. $x=-1+\cos t,\ y=3+\sin t;\ 0\le t<2\pi$
8. $x=2t,\ y=|t-1|;\ -\infty<t<\infty$
9. $x=t^2,\ y=t^3;\ -\infty<t<\infty$
10. $x=t^2+1,\ y=t^3-1;\ -\infty<t<\infty$

Eliminate the parameter and identify the graph of the parametric curve. If no interval is indicated, assume that $-\infty < t < \infty$.

11. $x = 1 + t,\ y = t$
12. $x = t,\ y = t^2 - 3$
13. $x = t,\ y = t^3 - 2t + 3$
14. $x = 2t - 3,\ y = 9 - 4t;\ 3 \le t \le 5$
15. $x = 2t - 4,\ y = 4t^2$
16. $x = 2\sin t,\ y = 2\cos t;\ 0 \le t < 2\pi$
17. $x = 3\cos t,\ y = 5\sin t;\ 0 \le t < 2\pi$
18. $x = 1 + 2\cos t,\ y = 2 + 3\sin t;\ 0 \le t < 2\pi$
19. $x = 1 + 3\cos t,\ y = -1 + 2\sin t;$ $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$
20. $x = \sqrt{t} + 2,\ y = \sqrt{t} - 2;\ t \ge 0$

21. An arrow leaves a compound bow with an initial velocity of 270 feet per second. The arrow is shot from 4 feet above the ground at an angle of $48°$ with the horizontal. Find the parametric equations that describe the position of the arrow. When does the arrow hit the ground? How far from its starting point does the arrow land?

22. A golfer at a driving range stands on a platform 12 inches above the ground and hits the ball with an initial velocity of 205 feet per second at an angle of $35°$ with the horizontal. There is a 32-foot-high fence 1200 feet away. Write the parametric equations that describe the path of golf ball. Will the ball fall short, hit the fence, or go over it?

23. A football kicked from the ground at an angle of $36°$ has an initial velocity of 65 feet per second. The goal post is 10 feet high and the ball is 105 feet from the goal post when it is kicked. Write the parametric equations that describe the path of the football. Will the football make it over the goal post?

24. A skeet is fired from the ground with an initial velocity of 112 feet per second at an angle of $25°$. Write the parametric equations that describe the path of the skeet. How long is the skeet in the air? How high does it go?

25. A baseball was hit with an initial velocity of 150 feet per second at an angle of $35°$ with the horizontal. The ball was hit at a height of 3 feet above the ground. Find the parametric equations that describe the position of the baseball. How long is the ball in flight? What is the total horizontal distance that it travels?

Geometry

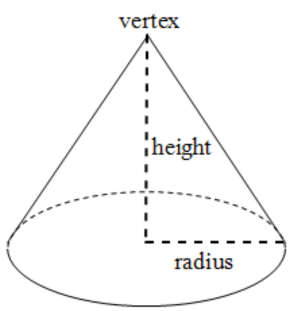
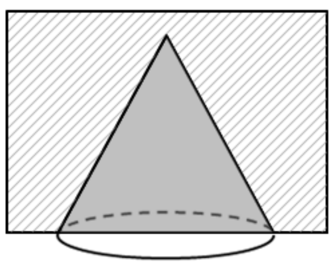
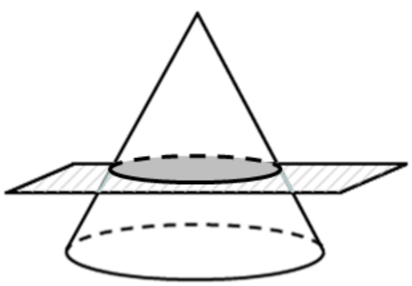
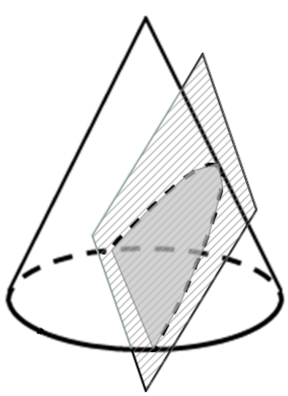
Unit 4 Cluster 7 (G.GMD.4): Two and Three-Dimensional Objects

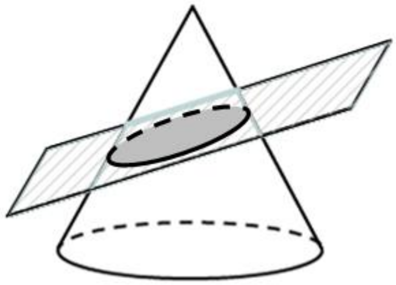
Cluster 7: Visualize relationships between two-dimensional and three-dimensional objects

4.7 Identify the two-dimensional shapes created from the cross-sections of three-dimensional objects.

4.7 Rotate two-dimensional objects and identify the three-dimensional objects created by the rotation.

Slicing or cutting through a three-dimensional figure with a plane can create a two-dimensional shape. For instance, slicing through a cone can create a triangle, circle, parabola, or ellipse.

	A **cone** is a three-dimensional figure that has a circle base and a vertex that is not in the same plane as the base. The height of the cone is the perpendicular distance between the vertex and the base.
	Slicing a cone through the vertex creates a triangle.
	Slicing a cone parallel to the base creates a circle.
	Slicing a cone diagonally, through the base, creates a parabola.

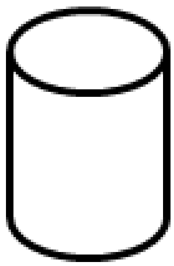
	Slicing a cone diagonally creates an ellipse.

Whenever a slice is made parallel to the base of the three-dimensional object then the two-dimensional cross-section created will be similar to the base. Additionally, the maximum number of sides that a two-dimensional cross-section can have is equal to the number of faces of the three-dimensional figure from which it is sliced.

Resources:
http://learnzillion.com/lessons/3314-visualize-crosssections-of-cones (video)
http://learnzillion.com/lessons/3386-visualize-crosssections-of-pyramids (video)
http://learnzillion.com/lessons/3469-visualize-crosssections-of-prisms (video)
http://learnzillion.com/lessons/3445-visualize-crosssections-of-cylinders (video)
http://www.learner.org/courses/learningmath/geometry/session9/part_c/ (applet)

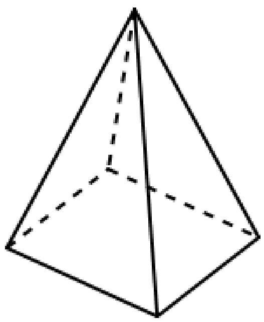
Practice Exercises A

Determine the two-dimensional cross-section that is created from each slice described.



Right Cylinder

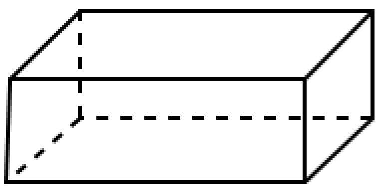
1. Horizontal slice.
2. Vertical slice.
3. Diagonal slice (not through a base).



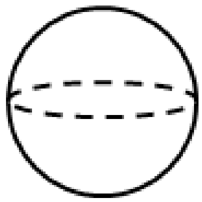
Square Based Pyramid

4. Horizontal slice.
5. Vertical slice through the vertex opposite the base.
6. Vertical slice not through the vertex opposite the base.
7. Diagonal slice through all four lateral sides and the base.

Right Rectangular Prism

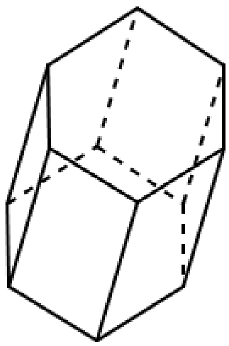


8. Horizontal slice.
9. Vertical slice.
10. A slice that cuts off a corner.
11. Diagonal slice through one base and all of the lateral sides.



Sphere

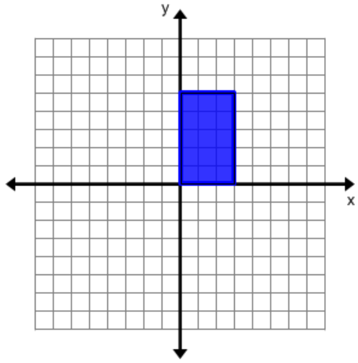
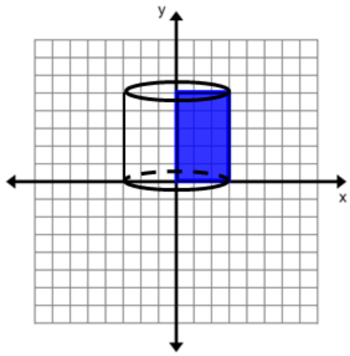
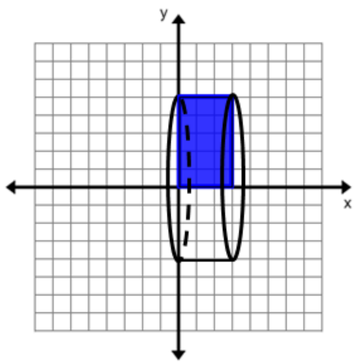
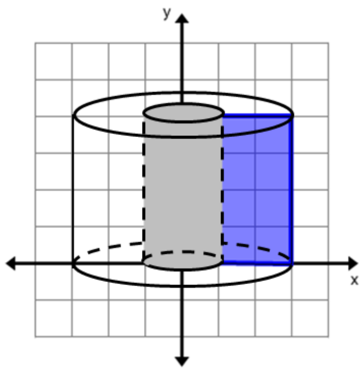
12. Horizontal slice.
13. Vertical slice.
14. Diagonal slice.



Hexagonal Prism

15. Horizontal slice.
16. Vertical slice.
17. A diagonal slice through all the lateral faces and one of the bases.
18. Can you make an octagon by slicing the shape? Why or why not?

Rotating a two-dimensional figure around an axis creates a three dimensional figure.

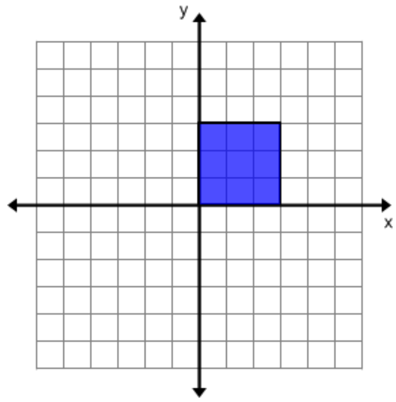
	Start with a rectangle that has a side on the each axis.
	Rotating around the *y*-axis creates a right circular cylinder with a height *y* and radius *x*.
	Rotating around the *x*-axis creates a right circular cylinder with a height *x* and radius *y*. Notice that the side perpendicular to the axis of rotation is flat, while the side parallel is curved.
	Rotating a rectangle that has only one side on an axis creates a cylinder with a hole in the middle or a doughnut.

Resources:
http://learnzillion.com/lessons/3488-predict-3d-results-of-rotating-simple-figures (video)

Practice Exercises B

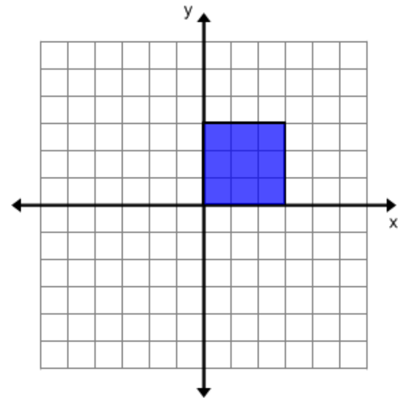
Sketch the result of rotating each shape around the given axis.

1.



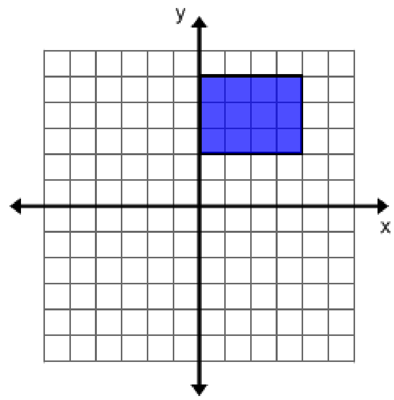
Around the *y*-axis

2.



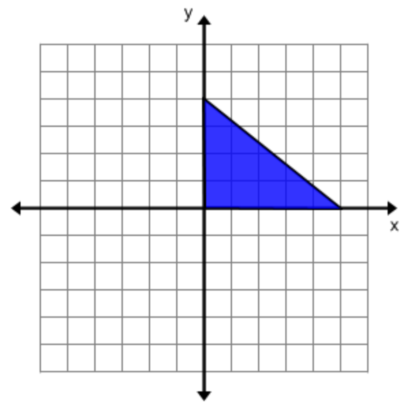
Around the *x*-axis

3.



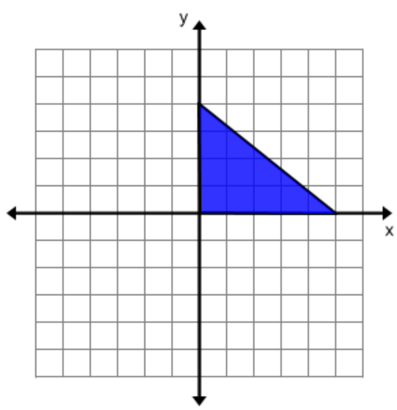
Around the *x*-axis

4.



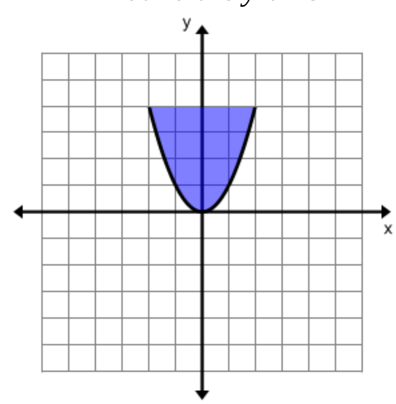
Around the *y*-axis

5.



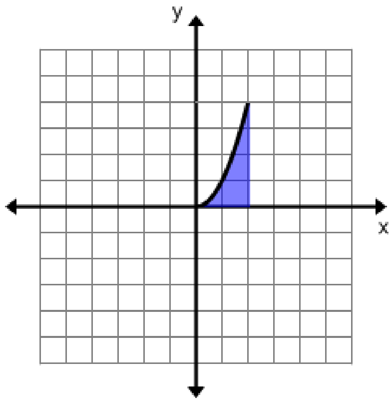
Around the *x*-axis

6.



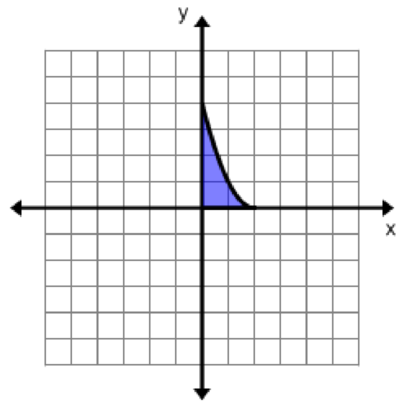
Around the *y*-axis

7.



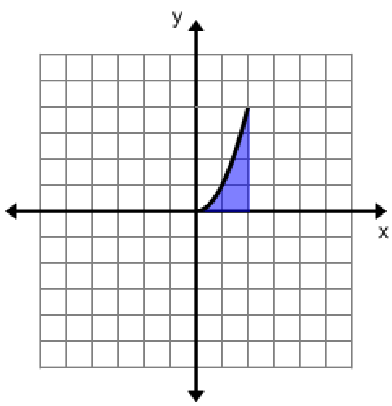
Around the *x*-axis

8.



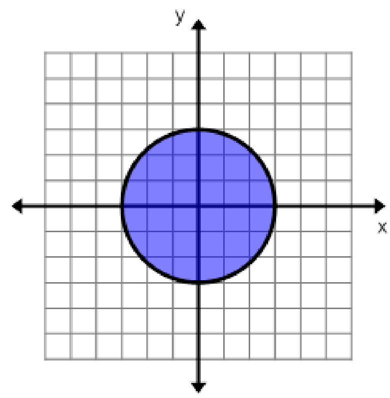
Around the *y*-axis

9.



Around the *y*-axis

10.



Around the *y*-axis

Unit 4 Cluster 8: G.MG.1, G.MG.2, & G.MG.3 Mathematical Modeling

Cluster 8: Apply geometric concepts in modeling situations

4.8 Use geometric shapes, their measures, and their properties to describe objects.

4.8 Apply geometric methods to solve design problem.

Using geometric shapes, you can estimate the areas and volumes of complex everyday objects.

Example 1:

How much surface area is painted on a regular yellow #2 pencil?

Looking at the end of a pencil before you sharpen it, you can see that a pencil is actually a hexagonal prism. Finding the lateral surface of the pencil will tell you how much of the surface is painted.	
$s = 4\text{ mm}$ $l = 17.1\text{ cm}$	Find the length of one side of the hexagonal base. And the length of the pencil not including the eraser.
$s = 4\text{ mm}$ $l = 171\text{ mm}$	Change each length to the same units of measure.
Painted Area $= (6)(4)(171)$ $= 4{,}104\text{ mm}^2$	There are six (6) lateral faces shaped like rectangles. Dimensions $4\text{ mm} \times 171\text{ mm}$

Example 2:

You are redecorating your house and want to paint an accent wall in your dining room. You will use a paint roller to paint the wall.

a. How much area will one rotation of the roller cover?

b. How many rotations will it take to cover a $8' \times 10'$ wall?

a. $C = \pi d$ $C = \pi(2.5)$ $C \approx 7.854\text{ inches}$	The diameter of a paint roller is 2.5 inches. The length is 9 inches. Find the circumference of the paint roller.
$A = Cl$ $A \approx (7.854)(9)$ $A \approx 70.686\text{ in}^2$ One rotation of the roller covers 70.686 in^2	Find the area that one rotation of the roller will cover in paint.

b. $(8)(12)=96$ inches $(10)(12)=120$ inches The wall is $96'' \times 120''$	Change the dimensions of the wall to inches.
$A=(96)(120)$ $A=11{,}520\text{ in}^2$	Find the area of the wall.
Number of rotations $\approx \dfrac{11520}{70.686}$ ≈ 162.975 You would need 163 full rotations to cover the wall with paint.	Divide the area of the wall by the area of the roller.

Example 3:
You need to buy bark to finish the landscaping of your yard. You have four flower beds measuring $90' \times 12'$, $3' \times 15'$, $10' \times 4'$, and $1.5 \times 22'$. The bark needs to be 6 inches deep. You can buy it by the bag for \$6.25 which covers 2 cubic feet or in bulk for \$60 which covers 1 cubic yard. Which is the least expensive way to purchase your bark?

$A=lw$ $A_1=(90)(12)$ $=1{,}080\text{ ft}^2$ $A_2=(3)(15)$ $=45\text{ ft}^2$ $A_3=(10)(4)$ $=40\text{ ft}^2$ $A_4=(1.5)(22)$ $=33\text{ ft}^2$	Find the area of each flower bed.
$A_1+A_2+A_3+A_4=1080+45+40+33$ $=1{,}198\text{ ft}^2$	Find the total area of the flower beds.
6 inches =0.5 feet Volume = $Area(depth)$ $=(1198)(0.5)$ $=599\text{ ft}^3$	Find the volume of bark you need, but first convert the depth to feet.
$\dfrac{599}{2} \approx 299.5$ You will need 300 2-cubic foot bags of bark	How many 2-cubic foot bags would you need? Divide the volume by 2.
$(300)(6.25)=1{,}875$ It would cost \$1,875 to buy 300 2-cubic foot bags of bark	Find the cost of buying 300 bags. Multiply the number of bags by \$6.25
1 cubic yard $=(3)(3)(3)$ $=27$ cubic feet	Convert 1 cubic yard to cubic feet

$\frac{599}{27} \approx 22.185$ You would need 23 loads of bulk material	Divide the volume of bark you need by 27.
$(23)(60) = 1380$ It would cost $1,380 to buy 23 loads of bulk bark	Find the cost of buying 23 loads of bulk bark. Multiply the number of loads by $60
You would save $495 by buying in bulk instead of in bags.	

Practice Exercises A

1. The base of a water bottle has diameter of 3.5 inches. The label wraps around the entire base of the bottle and has a height of 3.25 inches. What is the approximate surface area of the label?

2. You have a bookcase that is 78 inches high and 28 inches wide. It has 6 equally spaced shelves. You have textbooks that are $10\frac{7}{8}$ inches long and $1\frac{1}{8}$ inches wide. What is the maximum number of textbooks that can be stored in the bookcase?

3. You have decided to add a door to an existing wall. The wall is $10' \times 25'$. The door will be $3' \times 7'$. You want to finish the wall with brick and the size of a standard brick is $9'' \times 3''$. How many bricks will you need to complete the wall?

4. You are going to retile your shower. The dimensions of the shower are $32'' \times 32'' \times 73''$. You will be tiling 3 walls, the floor and the ceiling. It has a glass door but you will have to tile $6''$ below the door. You have three choices of tile: $12'' \times 12''$ square tile for $1.88 per square foot, $6'' \times 6''$ square tile for $3.16 per square foot, and $2'' \times 2''$ square tile for $3.65 per square foot. If you want to minimize the cost, which tile should you use, how many tiles will you need, and how much will it cost?

Challenge

You have been asked to design a parking lot for a new community center. It must have 200 parking stalls, 6 of which must be designated handicapped parking. All adjacent rows of stalls are perpendicular to each other. Regular stalls are 9 feet wide and 18 feet long. Handicapped stalls must be 8 feet wide and 18 feet long with a 5 foot access aisle on one side. Your parking lot needs 24 feet between parking rows. Keep in mind you need to minimize the square footage used for you parking lot.

VOCABULARY Density describes the amount of matter per volume of an object. Density can be compared by using the weight of a standard amount of the substance. The amount most scientists use is the cubic centimeter $\left(\text{cm}^3\right)$. For example, a cm^3 of helium weighs 0.0001785 grams, while cm^3 of air weighs 0.001205 grams. The helium is lighter than air which is why helium floats! Density can also describe the number of items in an area per square unit. For example population density measures the number of people in a given area.

Example 4:

You have a solid gold half sphere paper weight. It has a volume of approximately 17 cm^3. You want to replace it with a similar paper weight filled with sand. If the density of gold is about 19.32 g/cm^3 and the density of sand is about 2.5 g/cm^3, will the two paper weights weigh the same?

Gold Paper Weight: $19.32=\frac{\text{Mass}}{17}$ $(17)(19.32)=\text{Mass}$ $328.44 \text{ grams}=\text{Mass}$ Sand Paper Weight: $2.5=\frac{\text{Mass}}{17}$ $(17)(2.5)=\text{Mass}$ $42.5 \text{ grams}=\text{Mass}$ The two paper weights will not weigh the same.	Find the weight of each paper weight using $\text{Density}=\frac{\text{Mass}}{\text{Volume}}$ Solve for Mass

Example 5:

The 2010 census indicated that Utah had a population of 2,763,885. The area of Utah is 84,899 square miles. Find the population density of Utah.

$\text{Density}=\frac{\text{Population}}{\text{Area}}$	Use the density formula.
$\text{Density}=\frac{2{,}763{,}885}{84{,}899}$ $\approx 32.555 \text{ people per square mile}$	Substitute known values for the population and area. Divide.

Practice Exercises B

1. Texas has an area of 268,581 square miles and a population of 26,059,203 people. Vermont has an area of 9,614 square miles and a population of 626,011. Which state has the highest population density?

2. Strawberry Reservoir has a water surface of 26.817 square miles and Fish Lake has a water surface of 3.907 square miles. How many fish would you have to stock in both bodies of water in order for their fish population densities to be 500 fish per square mile?

3. You have a $2'' \times 4'' \times 8'$ cedar plank which weighs approximately 10.2 pounds and a $1'' \times 2'' \times 10'$ Oregon pine board which weighs approximately 4.6 pounds. Find the density of each piece of lumber and decide which wood is denser.

Statistics

Unit 1 Cluster 2 (S.IC.1): Statistical Inferences

Cluster 2: Understand and evaluate random processes underlying statistical experiments

1.2 Understand that statistics allows inferences to be made about population parameters based on a random sample from that population.

Suppose someone wanted to do a study about all sophomores, juniors, and seniors currently in high school in the United States. They might want to know the number of students who work after school, own a cell phone, or participate in extracurricular activities. Through statistical methods, we can gather and analyze information from a smaller sample population which allows us to make inferences about the much larger entire population. In Jordan School District there are currently 11,376 sophomores, juniors, and seniors in high school. It would be extremely costly and time consuming to interview every student; however, by taking a random sample of students we can calculate statistics which will allow us to draw conclusions about all 11,376 students in high school in Jordan School District.

VOCABULARY

A **population** consists of all people or items which we wish to describe or draw conclusions about. A **sample** is a small group of people or items taken from the larger population.

The population characteristic that we are interested in is called the **parameter of interest**. In the case of our high school example, the parameter of interest could be the number of students who work after school.

If it were possible to gather data from an entire population, then the parameter of interest would be the **population parameter**.

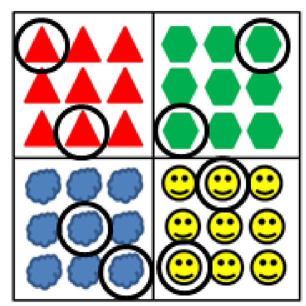
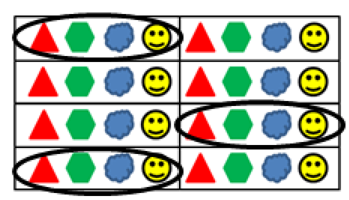
It is often difficult to gather data from an entire population so we use **statistics**, or data that we gather from a sample of the population, to make an **inference** or conclusion about the parameter of interest for the population.

In obtaining a sample from a population it is important to use **random sampling** to ensure the sample is representative of the population. Random sampling is a technique where a group is selected from the population. Each individual is chosen entirely by chance and each member of the population must have an equal chance of being included in the sample.

Example 1:
The Utah State Legislature wants to know what percentage of teen drivers text while they drive. They decide to survey 250 randomly selected teen drivers across the state. Identify a) the population, b) the sample population, and c) the parameter of interest.

a) the population is all teen drivers in the state of Utah
b) the sample is 250 teen drivers in the state of Utah
c) the parameter of interest is the percentage of teen drivers that text while they drive

Obtaining a random sample is not as simple as you might think. In fact, there are a few different methods for sampling. Some of the methods can be biased. A bias occurs when part of the population is overrepresented or underrepresented. For example, if you wanted to know how many students support the school's athletic programs, you wouldn't interview only the cheerleaders or students on a team, because they regularly attend athletic events and would be overrepresented in the study.

Sampling Methods	
In a **simple random sample** every member of the population has an equal chance of being selected to be part of the sample group. Drawing names from a hat is an example of this type of sampling. Another example would be assigning every member of the population a number and then using a random number table or generating random numbers through technology to randomly select members. The key is that you must have a list of all the members of the population.	
In a **systematic sample** it is assumed that the entire population is naturally organized in a sequential order. Using a random number generator, you select a starting point and then select every nth member to be part of the sample. For example, homes in a neighborhood are already in an order. You could randomly select a starting point and then select every third home.	
In a **stratified sample** members of the population that share the same characteristic are grouped together. Then, members of that subgroup are randomly selected to make up the sample group. Each member of the subgroup has an equal chance of being selected. There are times when the subgroups are not equal in size. When this happens, members are chosen in proportion to their actual percentages in the overall population. For example, if you wanted to study all high school students who are involved in extracurricular activities you would probably want to divide them into their particular extracurricular activity and then select randomly from those groups so that each extracurricular activity is represented in the sample. The football team would have more members than the basketball team so you would select more football players than basketball players to participate.	
In a **cluster sample** the population is divided into smaller groups that are representative of the entire population and then groups are randomly selected. For example, if you wanted to make inferences about your entire school, you could randomly select 1st periods to survey.	
In a **convenience sample** members are randomly selected from a population that is readily available. For example if you wanted to ask shoppers what they think of a local store, you would survey every 5th person who exits the store on a given day. This method of sampling has a bias because people who like to shop at this particular store are more likely to be at the store that day.	
In a **volunteer sample** members of the population self-select to be included in the sample. Filling out a survey and returning it is an example of a volunteer sample. This is prone to bias because generally people who respond have strong opinions about the topic while others who are more neutral may not respond at all. An example of a volunteer sample is when you buy a pair of shoes at your favorite shoe store and the cashier asks you to complete an online survey about your experience that day. You decide whether or not you want to complete the survey.	

Example 2:
The school newspaper wants to know the percentage of students who drive to school each day. For each method described below, determine what type of sampling method it is and justify whether or not the method is biased.

a. The newspaper staff posts signs all over the school asking students to take a short survey online.
b. The newspaper staff interviews every fifth person who walks into the school cafeteria.
c. The newspaper staff randomly selects 20 fifth periods to survey.

a. Volunteer Sample. This may or not be biased depending upon the question, how lengthy the survey is, and how difficult it is to complete.
b. Convenience Sample. This is biased because students who have cars are more likely to leave campus for lunch.
c. Cluster Sample. This is generally not biased.

Example 3:
You want to know if students at your school prefer fast food or sit-down restaurants. What would your survey question look like to eliminate any bias? Explain the sampling method you would use and why?

There are many sampling methods that would be appropriate and unbiased. This is one example.

The question: Where do you like to eat out?

Method: Systematic Sampling. The entire school population is generally known. I can obtain a list of the students in alphabetical order. I can randomly select one of the first ten students and then select every tenth student from there on. This eliminates bias because every student has an equal chance of being selected.

Practice Exercises A

For each situation described, identify a) the population, b) the sample population, and c) the parameter of interest.

1. An AP Government class wants to know the percentage of eligible voters in the state of Utah who voted in the most recent election. There are 1,938,249 people in Utah who are 18 and older. The class randomly looks at 15 state house districts and discovers that 50.5% of the eligible voters actually voted.

2. A local radio station has added an additional radio personality and is trying to determine what type of music to play during this person's air time. This time slot is geared towards teenage listeners. The station has decided to survey 300 randomly selected students from the ages of 13 to 19.

3. A health class wants to know the average amount of time Utahns over the age of 12 spend exercising each week. A sample of 1,200 randomly selected people, over the age of 12, across the state was surveyed.

For each method described below, determine what type of sampling method it is and justify whether or not the method is biased.

4. In order to determine the average composite score on the most recent ACT exam, students were divided into groups based on whether they were enrolled in remedial, regular, or honors language arts. Individual scores were randomly selected from each group.

5. Every third patron exiting the school musical was surveyed regarding their support for more funding of the arts.

6. A random number generator was used to assign students to demonstrate work in front of the class.

Project

7. Find a question of interest about the school population. Collect a random sample about the question of interest. Determine what inferences can be made about the population from that sample.

Unit 1 Cluster 2 (S.IC.2): Simulation

Cluster 2: Understand and evaluate random processes underlying statistical experiments

1.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.

VOCABULARY

The **theoretical probability** of an event occurring is the ratio of the number of favorable outcomes to the total number of outcomes, $P(\text{event})=\dfrac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$. It is what should happen in theory. For example, if we roll a die (a six-sided number cube) 60 times, in theory, the results should be 10 ones, 10 twos, 10 threes, etc. However, if we were to roll a die 60 times and calculate the probability based upon our data, the results may differ. The probability that we calculate from data we collect is the **experimental probability** of an event. It is defined to be the ratio of the number of favorable outcomes to the number of trials, $P(\text{event})=\dfrac{\text{number of favorable outcomes}}{\text{number of trials}}$. The **law of large numbers** says that the more trials you run in an experiment, the closer the experimental probability will get to the theoretical probability.

Simulations can use any random method of generating results as long as it fits the situation. A coin can be used to simulate an event with only two outcomes. A random number table can be used to simulate an event with 10 or 100 outcomes. A deck of cards can be used to simulate an event with 4, 13, or 52 outcomes. A random number generator, like a calculator, can be used to simulate any number of outcomes.

Example 1:
The school newspaper staff says it randomly picked 20 students to interview about the school's dress code. Fifteen of the students selected were boys. Does the number of boys selected cause you to question the selection process?

There are several simulations that you could use to model this situation.

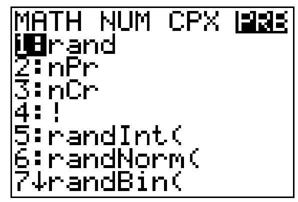
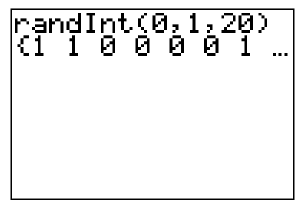
a. **flipping a coin:** Since there are only two outcomes: boys or girls, this is an acceptable method for simulating the selection. Let heads = boys and tails = girls. Flip a coin 20 times and record your data. Each set of 20 flips represents a single trial. Several trials are necessary in order for the simulation's statistic to be a good approximation.

Trial	**Heads Boys**	**Tails Girls**
1	9	11
2	12	8
3	11	9
4	8	12
5	13	7
6	6	14
7	7	13
8	10	10

9	13	7
10	9	11
11	7	13
12	13	7
13	12	8
14	13	7
15	9	11

From the 15 trials you can see that 15 heads and 5 tails never occurred. It is highly unlikely that the selection was random.

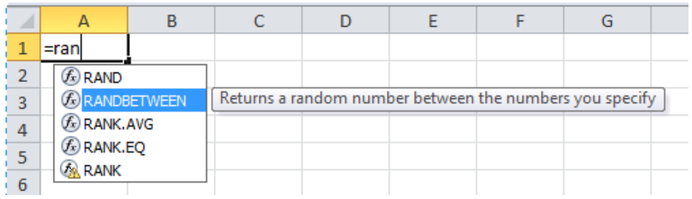
b. **using a random number generator:** the TI-83 and TI-84 calculators have a random number generator feature.

The random integer feature is in the MATH menu. Push  then use your arrow keys to arrow over to PRB. Option number 5 randInt(is the feature you want to use. Select it by pushing  or using your arrow keys to arrow down to 5 and pushing  .	
The syntax is randInt(lower number, upper number, number of numbers you want returned). In our simulation we can use 0 to represent boys and 1 to represent girls. Twenty students were chosen so we want to have 20 numbers returned. Use your arrow keys to see the numbers that are off the screen. By pushing  repeatedly you can do several trials.	

Trial	**0 Boys**	**1 Girls**
1	9	11
2	10	10
3	12	8
4	8	12
5	7	13
6	13	7
7	13	7
8	8	12
9	9	11
10	8	12
11	14	6
12	10	10
13	14	6
14	8	12
15	10	10

The data is consistent with flipping a coin. There were no occurrences of 15 and 5.

Microsoft Excel also has a random number generator feature that is similar to the calculator's random number feature.



The syntax is =RANDBETWEEN(lower number, upper number). Let 0 = boys and 1 = girls then in a cell type =RANDBETWEEN(0,1). You can copy the formula down the column so that it is in 20 cells. Copy the column several times and you have your simulation.

Trial	**0 Boys**	**1 Girls**
1	9	11
2	9	11
3	10	10
4	12	8
5	10	10
6	12	8
7	9	11
8	9	11
9	11	9
10	8	12
11	7	13
12	9	11
13	10	10
14	9	11
15	8	12

The data is consistent with flipping a coin and the calculator's random number generator. There were no occurrences of 15 and 5.

c. **random number table**: random number tables can be found on the internet. This particular table uses numbers 0 to 9 and groups them in clusters of 5. Let the even numbers including zero represent boys and the odd numbers represent girls. Use a random number generator to select a row to begin at. In this simulation, row 7 is where we will begin.

Random Number Table

16291	47751	28617	43266	75692	81384	25354	78664	35358	14658
93761	93658	15455	18589	64916	51584	17368	37478	53769	62767
76772	24458	49349	26977	55973	94643	77369	44195	68696	44356
64883	45331	43386	94778	35279	46898	63253	81918	63219	57955
64686	99491	32921	21687	27593	89286	56643	81317	94334	35217
67123	54977	86575	42722	91337	84614	76229	67517	23953	43454
39787	57814	17496	37277	43156	21483	44215	69351	11536	51665
87251	52193	94179	65383	26512	16476	56585	85955	25919	65346
51437	78564	57291	99419	15222	64582	62473	25812	26869	41256
69143	31827	31237	55455	47444	87593	97638	57597	68126	59583
72828	24116	42381	25452	14434	15131	53789	55711	75147	96269
86675	68946	62963	58266	54867	23988	97653	34312	31265	15965
46672	78525	64155	29222	47717	93568	65534	17878	97237	85737
24575	34765	61588	335411	57237	64314	51587	28797	46111	81988
42941	71328	39677	27853	25119	65448	84123	55469	46175	44911

The first group of numbers is: 3, 9, 7, 8, 7, 5, 7, 8, 1, 4, 1, 7, 4, 9, 6, 3, 7, 2, 7, and 7. There are 6 that are even or zero and 14 that are odd.

Trial	Even Boys	Odd Girls
1	6	14
2	9	11
3	6	14
4	9	11
5	7	13
6	6	14
7	13	7
8	11	9
9	7	13
10	7	13
11	13	7
12	4	16
13	11	9
14	13	7
15	6	14

The data is consistent with the other methods for simulation. There were no occurrences of 15 and 5.

There are many different ways to simulate a model. A die, colored marbles, or colored chips could also be used for simulations. You just need to make sure that the simulation is random and representative of the situation.

Practice Exercises A

Run a simulation of 50 trials to determine the probability indicated.

1. Given a 85% attendance rate, what is the probability that out of a class of 40 students exactly 34 will be present on any given day?

2. Lindsey has a 78% free-throw shooting record. What is the probability of having a 64% free-throw record in any given game?

3. Assume that 50% of the students enrolled in AP Calculus classes are male. In Ms. J's class, there are 24 students, 8 of which are female. What is the probability that this would happen by chance?

Unit 1 Cluster 3 (S.IC.3 & S.IC.6): Surveys, Experiments, Observations, and Evaluation of Reports

Cluster 3: Make inferences and justify conclusions from sample surveys, experiments, and observational studies

1.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

1.3 Evaluate reports based on data.

There are three main techniques for gathering a sample: sample surveys, experiments, and observational studies. Each of these methods has a purpose, advantages, and limitations. Randomization should occur in each of these methods.

VOCABULARY

When conducting a **survey** every member in the sample answers a set of questions.

Experiments require at least two groups. One group receives the trial treatment, while the other, sometimes called the control group, does not receive the treatment. At the end of an allotted period of time, the two groups are compared to determine if the treatment had an effect.

Observational studies require you to observe outcomes without interacting with any members of the sample.

Sample Surveys

The purpose of a sample survey is to gather information about the sample by means of a survey. There are several advantages to using a survey. Surveys are inexpensive and can collect a large amount of data representative of the population. They can be done in a variety of forms and about a variety of topics. Surveys also have the ability to focus only on the necessary information. However, surveys are flawed by non-responders since a survey is generally voluntary; people have the option not to participate. Additionally, people in a survey know that they are being studied and they may not be as honest in their responses as they would be if they were not being studied. Surveys are also open to interpretation and bias. Surveys can be written in a way that biases the responders. Also questions can be interpreted differently than intended by those responding to the survey.

Surveys can be administered with randomization methods, such as simple random sampling, cluster sampling, multistage sampling, stratified sampling, or systematic sampling all of which would ensure that the sample is random and representative of the overall population.

Experiments

The purpose of an experiment is to assign a treatment, using control over some of the conditions in order to gather data about the treatment's effectiveness. An experiment is the only way to establish causation. When an experiment is designed, all of the variables are controlled. This allows the experimenter to demonstrate that a change in one variable causes the change in another variable. There are drawbacks to experiments. They can be very expensive and time

consuming. Ethics may be questioned especially if animals or people are used in the experiment. Experiments must not intentionally harm any of the subjects. The attitude and behavior of those conducting the experiment can also affect the results.

It is imperative that randomization is used when assigning subjects to their treatment groups. Each group needs to be representative of the overall population.

Observational Studies

The purpose of an observational study is to observe subjects in their natural environment without their knowledge and without assigning treatments to the subjects. There are some advantages to using an observational study. It is simple and inexpensive to conduct. It provides deeper and richer information than a survey because the observer is seeing behavior firsthand and is able to observe the process not just the result. There are also some disadvantages. The results cannot prove causation nor can they be applied to the general population. It is only representative of those being studied. The results are subjective and open to interpretation by the observer. There may also be a question of ethics, especially if people are involved. People have a right to privacy and the observational study must not infringe upon the rights and expectations of people.

If you are doing the study in the present, you can randomize the individuals involved. If you are gathering data from past records, there is no chance for randomization.

Example 1:
Which type of study method is described in each situation? Should the sample statistics be used to make a general conclusion about the population?

a. Researchers randomly choose two groups from 20 volunteers. Over a period of 6 weeks, one goup works on a computer for an hour right before going to sleep, and the other does not. Volunteers wear monitoring devices while sleeping, and researchers record their quality of sleep.

b. Students in an elementary class observe the growth of some newly hatched chickens.

c. Market researchers want to know if people like the new store at the local mall. They ask every fourth person who enters the mall if they like the new store.

Answer:

a. This is an experiment. There are two groups: a treatment group and a control group. There are very few participants in the study so it is not a good idea to generalize this to the entire population.

b. This is an observational study. Observational studies can't be generalized to the entire population. There may be variables that are not controlled such as amount of food available, the climate, etc.

c. This is a survey. The results of the survey can only be applied to the population that shops at the local mall. It wouldn't account for the opionion of those who do not shop at the local mall.

Practice Exercises A

Which type of study method is described in each situation? Should the sample statistics be used to make a general conclusion about the population?

1. A numbered list of students is generated from the school database. Students are randomly selected from the list by using a random number generator. Information for every student is entered into the database, and each student has an equally likely chance of being selected. The students selected are asked how much allowance money they are given each week for doing chores at home.

2. The owner of a bakery collects data about the types of cupcakes that are purchased so she can make cupcakes accordingly. She records they type of cupcake purchased by every other person each day for three weeks.

3. A botanist tests a new breed of plant by planting seeds from the new breed of plant and the traditional breed of plant in the same soil and the same location. He ensures that both types receive the same amount of water and plant food. He records the growth rates of both types of plants.

4. Every day for two weeks, a student records the number of her classmates who are late to class.

5. A local grocery store selects 350 customers from a list of 1500 new customers in the past year to mail a questionnaire. There are 245 customers who return the questionnaire.

6. A teacher wants to know if playing classical music while a class works on a test will improve their scores on the test. She uses two class periods of equal size (35 students in each class) and equal baseline test data. For an entire semester, she plays classical music while one class is testing and plays no music while the other class is testing.

Evaluate Reports

Statistics are reported everywhere. It is important to look at any statistical reporting critically and evaluate the content for its validity in regards to the general population. Here are some things to consider when evaluating any report containing statistical information.

- Sampling method
- Study type
- Population of interest
- Bias
- Sample size
- Study duration

Practice Exercise B

1. Use the criteria above to evaluate the reports found at the links below.

   Dan Jones and Associates (Utah based company)
   http://cppa.utah.edu/_documents/publications/governance/hb-40-final-report.pdf

   Gallup Student Polls (National Company)
   http://www.gallupstudentpoll.com/159221/gallup-student-poll-overall-scorecard-fall 2012.aspx

Unit 1 Cluster 1 (S.ID.4): Normal Distribution

Cluster 1: Summarize, represent, and interpret data on a single count or measurement variable

1.1 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages.

1.1 Recognize that there are data sets for which such a procedure is not appropriate.

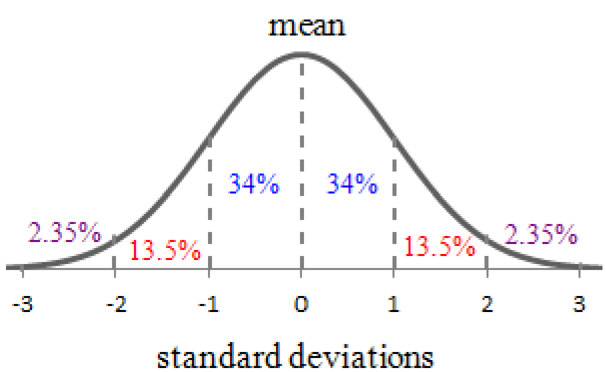
1.1 Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

VOCABULARY

Data that is unimodal, symmetric, and with no outliers is said to be normally distributed. A **normal distribution** is bell shaped with mean, μ, at the center of the curve.

In a normal distribution,

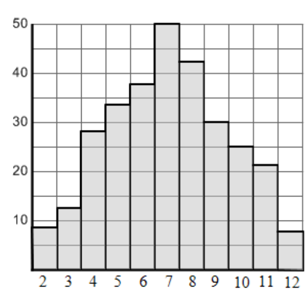
- 68% of the data fall within one standard deviation, $\pm\sigma$, of the mean
- 95% of the data fall within two standard deviations, $\pm 2\sigma$, of the mean
- 99.7% of data fall within three standard deviations, $\pm 3\sigma$, of the mean.



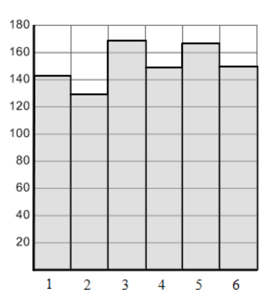
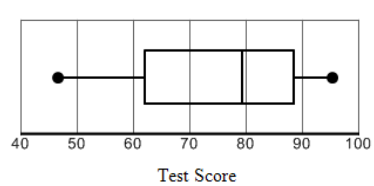
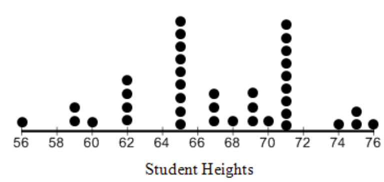
Nearly all data lie within three standard deviations from the mean. This is known as the **empirical rule**.

The area under a normal curve is always 1. When calculating population percentages, the value will be less than 1.

Normal Distribution



Not all data sets are distributed normally. Notice the examples below. Some data sets are uniform, left or right skewed, or bimodal.

Uniform Distribution	Left Skewed	Bimodal
		

Example 1:
ACT test scores are approximately normally distributed. One year the scores had a mean of 21 and a standard deviation of 5.2.

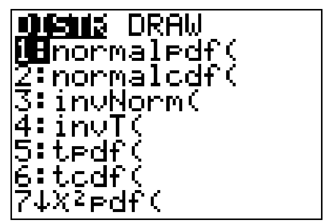
a. What is the interval that contains 95% of scores?
b. What percentage of ACT scores is less than 25.2?
c. What percentage of ACT scores is between 28 and 36?

a. $\mu \pm 2\sigma$ $21 \pm 2(5.2)$ 21 ± 10.4 $21-10.4=10.6 \qquad 21+10.4=31.4$ The interval that contains 95% of the scores is from 10.6 to 31.4.	The interval containing 95% of the scores would be $\mu \pm 2\sigma$. $\mu = 21$, $\sigma = 5.2$

b. Since the score 25.2 is not quite one standard deviation away from the mean, $x-\mu = 25.2-21=4.2$, the empirical rule cannot be used to calculate the percentage. You will need to use a graphing calculator or a spreadsheet to calculate the proportion.

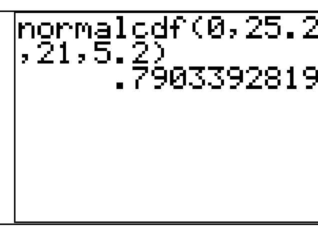
Using a TI-83 or TI-84 Graphing Calculator

The distribution features are found by pushing 2ND DISTR VARS. A menu like the one at the right should appear. Option 2, normalcdf(is the feature that you want to use. This feature is the normal cumulative distribution function. It will calculate the percentage of data that fall between two numbers. Select option 2 by pushing L2 Z 2 or by using your arrow keys to arrow down to 2 and pushing ENTRY SOLVE ENTER.



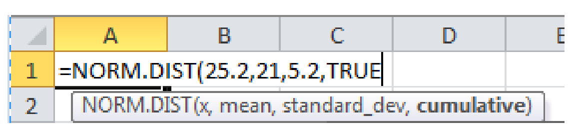
The syntax required for this feature is normalcdf(lower bound, upper bound, mean, standard deviation). In the case of our example, it would be normalcdf(0, 25.2, 21, 5.2).

Approximately 79% of the scores are below 25.2.



Using an Excel Spreadsheet

In a cell start an equation by typing an equal sign and then type NORM.DIST. The syntax for entering information is NORM.DIST(value, mean, standard deviation, cumulative). By stating that the cumulative part is TRUE the entire percentage up to the value specified will be calculated.



In the case of our example, you should type =NORM.DIST(25.2, 21, 5.2, TRUE) and then push enter.

Approximately 79% of the scores are below 25.2.

	A
1	0.790366

c. Approximately 9% of the scores are between 28 and 36.

Use a calculator to find the percentage. Use normalcdf(.

```
normalcdf(28,36,
21,5.2)
       .0871669871
```

You could also use Excel, but you would have to do a little extra work. You would need to find the percentage that scored less than 28 and the percentage that scored less than 36 and then find the difference between the two.

	A
1	0.910874
2	0.998041
3	0.087167

Practice Exercises A

1. The mathematics portion of the SAT has a mean score of 500 and a standard deviation of 100.
    a. What is the interval that contains 99.7% of scores?
    b. What percentage of SAT scores is greater than 650?
    c. What percentage of SAT scores is between 325 and 615?

2. Americans consume 16.5 pounds of ice cream per year with a standard deviation of 3.25 pounds.
    a. What is the interval that contains 68% of the pounds consumed each year?
    b. What percentage of pounds consumed is less than 10 pounds?
    c. What percentage of pounds consumed is between 5 pounds and 11pounds?

3. The average height of a NBA basketball player is 79 inches with a standard deviation of 3.89 inches.
    a. What is the interval that contains 95% of the heights?
    b. What percentage of the heights is greater than 81 inches?
    c. What percentage of the heights is between 73 inches and 77 inches?

Unit 1 Cluster 3 (S.IC.4): Margin of Error

Cluster 3: Make inferences and justify conclusions from sample surveys, experiments, and observational studies

1.3 Use data from a sample survey to estimate a population mean or proportion.

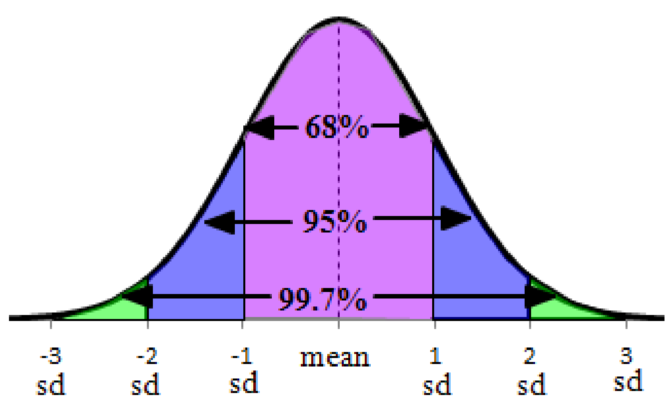
1.3 Develop a margin of error through the use of simulation models for random sampling.

When we use a simulation to model an event, it is only an approximation of the population parameter. If we were to run the simulation numerous times, each result would be slightly different. However, it is possible to give an interval that the population parameter falls within by finding a margin of error.

A margin of error is not a mistake; rather it refers to the expected range of variation in a survey or simulation if it were to be conducted multiple times under the same procedures. The margin of error is based on the sample size and the confidence level desired. The interval that includes the margin of error is called the **confidence interval** and is usually computed at a 95% **confidence level**. A confidence level of 95% means that you can be 95% certain that the actual population parameter falls within the confidence interval.

For example, the student body officers at your school conducted a survey to determine whether or not a majority of the students dislike the music played in the hall during class changes. They randomly interviewed students and determined that 52% of the students disliked the music with a margin of error of 3% that was calculated at a 95% confidence level. Can the student body officers say for certain that over half of the student body dislikes the music? The confidence interval for this situation is $52\% \pm 3\%$ or 49% to 55%. The confidence level of 95% allows us to say that we are 95% confident that the true percentage is between 49% and 55%. However, it is plausible that the percentage of students who dislike the music is less than 50%. Therefore, it is probably not wise to approach the administration about changing the music just yet.

When calculating the means from several trials of a simulation, the results are normally distributed. Recall that in a normal distribution 68% of the data falls within 1 standard deviation of the mean, 95% of the data falls within two standard deviations of the mean, and 99.7% of data falls within three standard deviations of the mean. We can use the fact that 95% of the data is within 2 standard deviations of the mean to find a margin of error with a 95% confidence level.



VOCABULARY

The **margin of error** accounts for the variation in results if the study or simulation were to be conducted multiple times under the same conditions. It does account for the random selection of individuals but it does not account for bias.

A **confidence interval** provides a range of plausible values for a population parameter. It can be found by using the sample statistic and the margin of error for the confidence level desired, i.e. $\text{sample statistic} \pm \text{margin of error}$.

The **confidence level** determines how likely it is that the actual population parameter falls on the confidence interval that was calculated using the sample statistic and the margin of error.

Calculating the Margin of Error

The following formulas can be used to approximate the margin of error with a 95% confidence level:

1. If you are given a sample proportion, then the margin of error can be approximated by:

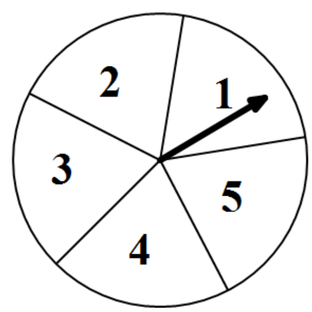
   $\text{margin of error} = 2 \cdot \sqrt{\dfrac{\rho(1-\rho)}{n}}$, where ρ is the sample proportion and *n* is the sample size.

2. If you are given sample mean and standard deviation, then the margin of error can be approximated by: $\text{margin of error} = 2 \cdot \dfrac{s}{\sqrt{n}}$, where *s* is the sample standard deviation and *n* is the sample size.

Example 1:

A spinner like the one shown at the right was spun 30 times and the number it landed on was recorded as shown below.

1	2	3	4	5
~~||||~~ |	||||	~~||||~~ |	||||	~~||||~~ ||||



For each situation, find the sample proportion, the margin of error for a 95% confidence level, the 95% confidence interval for the population proportion, and determine if the theoretical proportion would be within the confidence interval found.

a. the probability of the spinner landing on 2
b. the probability of the spinner landing on 3
c. the probability of the spinner landing on 5

Answer:
The theoretical probability for each number will be the same. It can be found by using the formula:

$P(E) = \dfrac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$. In the case of the spinner, there is one number for the favorable outcome and 5 numbers for the total number of outcomes, therefore, the theoretical probability will be $P(E) = \dfrac{1}{5} = 0.2$.

a. $P(2)=\frac{\text{number of favorable outcomes}}{\text{total number of trials}}=\frac{4}{30}\approx 0.133$	The spinner landed on 2 four times out of 30 trials.
$\text{margin of error}=2\cdot\sqrt{\frac{0.133(1-0.133)}{30}}$ $\text{margin of error}\approx 0.124$	We have a sample statistic, so we will use the formula $\text{MOE}=2\cdot\sqrt{\frac{\rho(1-\rho)}{n}}$ to approximate the margin of error.
0.133 ± 0.124 $0.133-0.124=0.009$ $0.133+0.124=0.257$ The confidence interval is from 0.009 to 0.257.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
The theoretical probability of 0.2 would fall in the confidence interval from 0.009 to 0.257.	

b. $P(3)=\frac{\text{number of favorable outcomes}}{\text{total number of trials}}=\frac{6}{30}=0.2$	The spinner landed on 3 six times out of 30 trials.
$\text{margin of error}=2\cdot\sqrt{\frac{0.2(1-0.2)}{30}}$ $\text{margin of error}\approx 0.146$	We have a sample statistic, so we will use the formula $\text{MOE}=2\cdot\sqrt{\frac{\rho(1-\rho)}{n}}$ to approximate the margin of error.
0.2 ± 0.146 $0.2-0.146=0.054$ $0.2+0.146=0.346$ The confidence interval is from 0.054 to 0.346.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
The theoretical probability of 0.2 would fall in the confidence interval from 0.054 to 0.346.	

c. $P(5)=\frac{\text{number of favorable outcomes}}{\text{total number of trials}}=\frac{9}{30}=0.3$	The spinner landed on 5 nine times out of 30 trials.
$\text{margin of error}=2\cdot\sqrt{\frac{0.3(1-0.3)}{30}}$ $\text{margin of error}\approx 0.167$	We have a sample statistic, so we will use the formula $\text{MOE}=2\cdot\sqrt{\frac{\rho(1-\rho)}{n}}$ to approximate the margin of error.
0.3 ± 0.167 $0.3-0.167=0.133$ $0.3+0.167=0.467$ The confidence interval is from 0.133 to 0.467.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
The theoretical probability of 0.2 would fall in the confidence interval from 0.133 to 0.467.	

Example 2:
A fast food restaurant manager wanted to determine the wait times for customers in line. He timed the customers chosen at random.

Wait Time in Minutes		
6.4	9.3	3.9
4.3	6.4	4.8
6.7	3.7	5.4
4.0	3.3	5.9
4.5	8.1	2.6
2.9	4.4	3.1
6.0	5.5	5.9
3.6	8.0	3.7
7.7	8.1	9.4
9.9	9.4	2.6

a. Find the mean and standard deviation for the sample. (Round to the nearest tenth)
b. Approximate the margin of error for a 95% confidence level and round to the nearest tenth.
c. Find the 95% confidence interval.
d. Interpret the meaning of the interval in terms of wait times for customers.

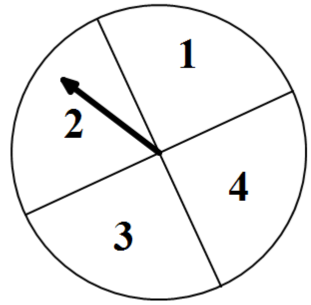
Answer:

a. The sample mean, $\bar{x}$, is 5.7. The sample standard deviation, *S*, is 2.2.	Enter the data in a graphing calculator and run the one-variable statistics.
b. $\text{margin of error} = 2 \cdot \frac{s}{\sqrt{n}}$ $\text{margin of error} = 2 \cdot \frac{2.2}{\sqrt{30}}$ $\text{margin of error} \approx 0.8$	We have a sample mean and standard deviation so use the formula $\text{MOE} = 2 \cdot \frac{s}{\sqrt{n}}$ to approximate the margin of error.
c. 5.7 ± 0.8 $5.7 - 0.8 = 4.9$ $5.7 + 0.8 = 6.5$ The confidence interval is from 4.9 minutes to 6.5 minutes.	The confidence interval is found by adding and subtracting the margin of error from the sample statistic.
d. We can say with 95% confidence that the actual average wait time for a customer at the fast food restaurant is between 4.9 minutes and 6.5 minutes.	

Practice Exercises A

1. A spinner like the one shown was spun 40 times and the number it landed on was recorded as shown below.

1	2	3	4
𝍸 𝍷𝍷𝍷𝍷	𝍸	𝍸 𝍸 𝍷𝍷	𝍸 𝍸 𝍷𝍷𝍷𝍷



For each situation, find the sample proportion, the margin of error for a 95% confidence level, the 95% confidence interval for the population proportion, and determine if the theoretical proportion would be within the confidence interval found.

   a. the probability of the spinner landing on 1
   b. the probability of the spinner landing on 2
   c. the probability of the spinner landing on 4

2. A consumer research group tested battery life of 36 randomly chosen cell phones to establish the likely battery life for the population of the same type of cell phone.

Battery Life in Hours			
55.4	63.3	72.7	70.6
50.2	85.4	85.2	83.2
72.0	69.5	65.4	65.1
55.7	73.1	47.9	72.9
55.3	58.6	81.1	58.5
64.0	83.7	73.0	74.7
80.0	73.9	75.4	58.9
61.3	69.8	83.3	61.2
63.0	63.1	85.0	57.6

   a. Find the mean and standard deviation for the sample. (Round to the nearest tenth)
   b. Approximate the margin of error for a 95% confidence level and round to the nearest tenth.
   c. Find the 95% confidence interval.
   d. Interpret the meaning of the interval in terms of battery life for this type of cell phone.

3. In a poll of 650 likely voters, 338 indicated that they planned to vote for a particular candidate.
   a. Find the sample proportion.
   b. Approximate the margin of error for a 95% confidence level.
   c. Find the 95% confidence interval.
   d. Interpret the meaning of the interval in terms of the election.

4. A recent school poll showed that 47% of respondents favored Willy B. Smart for student body president, while 51% favored Flora Bunda, with a margin of error of 4% for each poll. Can a winner be determined from the poll?

Unit 1 Cluster 3 (S.IC.5): Compare Two Treatments

Cluster 3: Make inferences and justify conclusions form sample surveys, experiments, and observational studies

1.3 Use data from a randomized experiment to compare two treatments.

1.3 Use simulations to decide if differences between parameters are significant.

Doing an experiment is the only way to prove causation. Experiments include at least one treatment group and a control group. A control group is made up of experimental units that do not receive the treatment. By comparing the results from the treatment group and the control group, we can decide if the differences are enough to convince us that the treatment was effective. We call these "statistically significant" results. Using a technique called **resampling** we can compare the results of two treatments and determine if the treatment's effect was by chance or statistically significant. Resampling uses existing sample data from an experiment. It is assumed that the treatment has no discernible effect on the units and therefore a reassignment of units between the two groups would result in the same statistic. To perform a resampling, you combine all of the results in one data set. You conduct a simulation with several trials that uses random sampling to redistribute the data into new treatments groups. The data from each trial is compiled into a plot. The original data is then compared to the simulation data.

Example 1:
Seventy students were selected to participate in an experiment to test the effectiveness of a new homework policy. Thirty-five of the students were randomly selected to be in a group that received the new homework policy and thirty-five of the students were selected to be in a group that did not receive the new homework policy. At the end of the study, the grades of 27 students who received the new homework policy increased compared to 20 students who were in the group with no change in the homework policy. Determine if the results are significant.

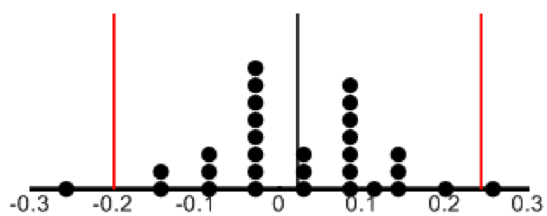
Answer:
We were not given the data, but we have the proportion of students who improved for each group. Combining this information we have 47 students out of a total of 70 students who improved. We are going to assume that this would happen regardless of whether or not the students received the new homework policy.

To simulate the situation, 47 red marbles were used to represent the students who improved and 23 blue marbles were used to represent the students who did not improve. Thirty-five marbles were drawn without replacement and the number of red marbles drawn was recorded. Thirty trials were done. The results are displayed below.

Trial	New Homework Policy		Old Homework Policy		Difference Between Proportions
	Red Marbles	Proportion	Red Marbles	Proportion	
1	25	0.714	22	0.629	0.085
2	21	0.600	26	0.743	-0.143
3	23	0.657	24	0.686	-0.029
4	28	0.800	19	0.543	0.257
5	22	0.629	25	0.714	-0.085
6	23	0.657	24	0.686	-0.029
7	25	0.714	22	0.629	0.085
8	23	0.657	24	0.686	-0.029
9	25	0.714	22	0.629	0.085

10	25	0.714	22	0.629	0.085
11	24	0.686	23	0.657	0.029
12	25	0.714	22	0.629	0.085
13	23	0.657	24	0.686	-0.029
14	24	0.686	23	0.657	0.029
15	26	0.743	21	0.600	0.143
16	23	0.657	24	0.686	-0.029
17	23	0.657	24	0.686	-0.029
18	21	0.600	26	0.743	-0.143
19	24	0.686	23	0.657	0.029
20	26	0.743	21	0.600	0.143
21	23	0.657	24	0.686	-0.029
22	25	0.714	21	0.600	0.114
23	19	0.543	28	0.800	-0.257
24	23	0.657	24	0.686	-0.029
25	22	0.629	25	0.714	-0.085
26	25	0.714	22	0.629	0.085
27	27	0.771	20	0.571	0.200
28	25	0.714	22	0.629	0.085
29	22	0.629	25	0.714	-0.085
30	26	0.743	21	0.600	0.143

The dot plot below graphs the difference between the proportion of the new policy and the old policy. The two vertical lines on the outside edge represent 2 standard deviations below and above the mean. The mean for the difference between proportions in the simulation is 0.0217.



In our original test groups the proportions were:

$$\text{Treatment group: } \frac{\text{number of students who improved}}{\text{total number of students}} = \frac{27}{35} \approx 0.771$$

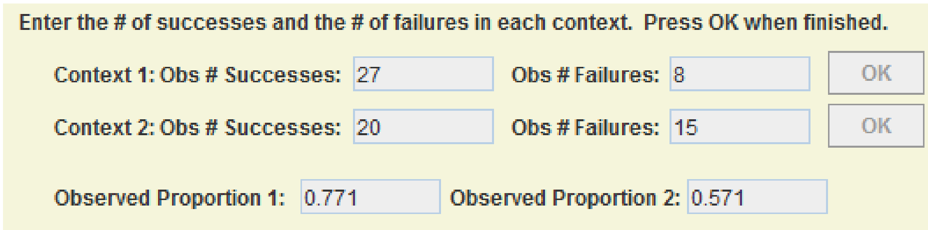
$$\text{Control group: } \frac{\text{number of students who improved}}{\text{total number of students}} = \frac{20}{35} \approx 0.571$$

The difference was: $\text{Proportion new } - \text{ Proportion old} = 0.771 - 0.571 = 0.2$.

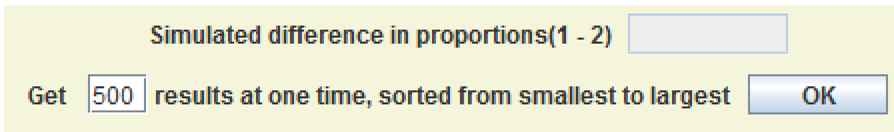
We need to determine if a difference of 0.2 is statistically significant. Based on our simulation, it is not outside the interval that contains 95% of the data (95% of all data lies within two standard deviations of the mean in a normal distribution), therefore it is not statistically significant.

You can use a computer simulation to do more trials. The simulation can be found at: http://bcs.whfreeman.com/sris/#730892__752214__. For this simulation we are finding the difference in proportions. Use the applet that says "Difference in Proportions." Context 1

represents the students who received the new homework policy (27 successes and 8 failures) and context 2 represents the students who did not receive it (20 success and 15 failures).



Down below you can type in how many results you want the computer to do at one time. For this simulation 500 was selected.



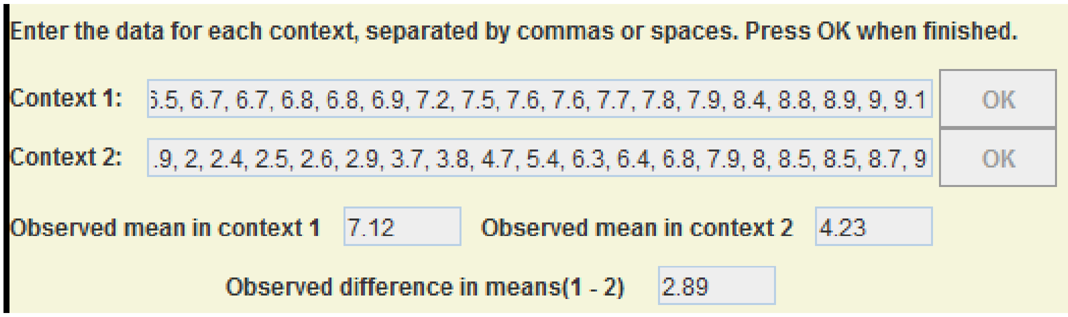
The data was exported to an excel spreadsheet. The mean was -0.00057 and the standard deviation was 0.108825. The interval containing 95% of the data is from -0.218 to 0.217. Our sample difference of 0.2 is still within this interval so it is not significant.

Example 2:
Fifty students were selected to participate in an experiment to test the effectiveness of a new shampoo that claimed to have an ingredient to make hair shinier. Twenty-five of the students were randomly selected to receive the new shampoo and twenty-five of the students were selected to receive the same shampoo but without the new ingredient. After six weeks of using the shampoo, the two groups were analyzed by hair shininess experts. The sample data is displayed in the table below.

Group	Level of shininess compared to before use of shampoo
Shampoo with ingredient	5.3, 5.5, 5.5, 5.7, 5.8, 5.9, 6.3, 6.5, 6.7, 6.7, 6.8, 6.8, 6.9, 7.2, 7.5, 7.6, 7.6, 7.7, 7.8, 7.9, 8.4, 8.8, 8.9, 9, 9.1
Shampoo without ingredient	0.3, 0.5, 0.6, 0.6, 0.6, 1.2, 1.9, 2, 2.4, 2.5, 2.6, 2.9, 3.7, 3.8, 4.7, 5.4, 6.3, 6.4, 6.8, 7.9, 8, 8.5, 8.5, 8.7, 9

The mean change in shininess of the group who used the new ingredient shampoo is 7.116 and the mean change in shininess of the group who used the regular shampoo is 4.232. Using a computer simulation with 500 trials, determine if this a significant difference.

Answer:
Go to the website http://bcs.whfreeman.com/sris/#730892__752214__. Then use the statistics applet that says "Difference in Means." Enter the data for the new ingredient shampoo group in context 1 and the data for the regular shampoo group in context 2. Then at the bottom have the simulation run 500 trials.



The mean of the differences in the simulation is -0.0287. The standard deviation of the simulation differences is 1.595. The interval that contains 95% of the data is from -1.624 to 1.566. The sample difference we began with had a difference in means of 2.89. This is beyond the interval so it is significant.

Practice Exercises A

1. Seventy-two students were selected to participate in a study to determine if computerized testing distracts students enough to lower test scores. All students were given a baseline test using paper and pencil. Thirty-six of the students were randomly selected to be in a group that used an electronic device to take a post-test. The other thirty-six students took their test using paper and pencil. After completing the post-test, the scores of twenty students using the computer were lower than the baseline test, while the scores of ten students using paper and pencil were also lower. Design a simulation to determine if the results are significant.

2. A farmer wants to know if a new enriched top soil will produce better corn plants. The farmer has 20 fields. He randomly assigns 10 fields to receive the enriched top soil and 10 fields to receive the same top soil he has used in the past. At harvest time, fields that received the enriched top soil had corn plants that had grown to mean heights of 6.5 feet, 7.9 feet, 8 feet, 7.2 feet, 7.6 feet, 6.6 feet, 8 feet, 6.3 feet, 7.8 feet, and 6.8 feet while the plants with no enriched soil grew to mean heights of 6.8 feet, 6.8 feet, 6.9 feet, 6.4 feet, 6.8 feet, 6.8 feet, 6.5 feet, 6.4 feet, 6.4 feet, and 7 feet. Using a computer simulation with 500 trials, determine if this a significant difference.

Selected Answers

Secondary Mathematics 3 Answers

Unit 2 Cluster 4 (A.APR.1)

Practice Exercises A (pg. 5)

1. $2x^2 - 3x - 8$
3. $-3x^2 + 6$
5. $2n^3 + 4n^2 - 8$
7. $-3x^2 - x + 2$
9. $9x^3 + x^2 + 6x - 10$
11. $x^2 - 4x + 10$

Practice Exercises B (pg. 7−8)

1. $-30x^2 - 21x + 18$, polynomial
3. $70x^2 + 55x + 10$, polynomial
5. $-36x^2 - 77x - 40$, polynomial
7. $4x^2 + 28x + 49$, polynomial
9. $25x^6 - 10x^3 + 1$, polynomial
11. $x^3 + 2x^2 - 5x + 12$, polynomial
13. $10x^3 + 43x^2 + 30x + 7$, polynomial
15. $27x^3 + 78x^2 + 61x + 10$, polynomial
17. $x^4 + 2x^3 - x^2 - 2x - 3$, polynomial
19. $2x^4 - 6x^3 - 78x^2 - 42x - 4$, polynomial
21. $5y^4 + 13y^3 - 5y^2 - y - 12$, polynomial
23. $-20x^4 - 42x^3 - 47x^2 - 39x - 6$, polynomial

You Decide (pg. 8)

Polynomials are closed under addition, subtraction and multiplication. All of the answers to Practice Exercises B were polynomials.

Unit 2 Cluster 5 (A.APR.2, A.APR.3, and F.IF.7c)

Practice Exercises A (pg. 10)

1. a. $f(-1) = -24$, not a factor
   b. $f(7) = 0$, factor
   c. $f(2) = 15$, not a factor

3. a. $f(-2) = 0$, factor
   b. $f(2) = 0$, factor
   c. $f(-3) = 0$, factor

Practice Exercises B (pg. 14)

1. $\lim\limits_{x\to-\infty} f(x) \to -\infty$, $\lim\limits_{x\to\infty} f(x) \to +\infty$

3. $\lim\limits_{x\to-\infty} f(x) \to +\infty$, $\lim\limits_{x\to\infty} f(x) \to -\infty$

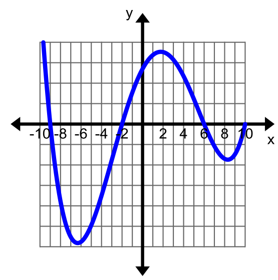
5. $\lim\limits_{x\to-\infty} f(x) \to -\infty$, $\lim\limits_{x\to\infty} f(x) \to -\infty$

Practice Exercises C (pg. 15)

1. 4
3. 5
5. 6

Practice Exercises D (pg. 16)

1.

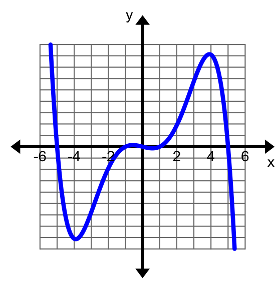


$(-9,0)$, $(-2,0)$, $(6,0)$, $(10,0)$

$\lim\limits_{x\to-\infty} f(x) \to +\infty$

$\lim\limits_{x\to\infty} f(x) \to +\infty$

3.

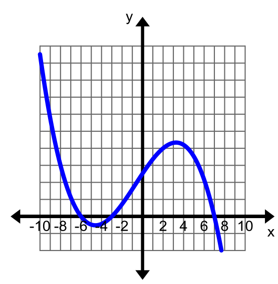


$(-5,0)$, $(-1,0)$, $(0,0)$, $(1,0)$, $(5,0)$

$\lim\limits_{x\to-\infty} f(x) \to +\infty$

$\lim\limits_{x\to\infty} f(x) \to -\infty$

5.

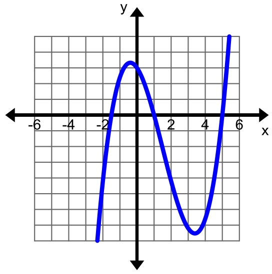


$(-6,0)$, $(-3,0)$, $(7,0)$

$\lim\limits_{x\to-\infty} f(x) \to +\infty$

$\lim\limits_{x\to\infty} f(x) \to -\infty$

7.



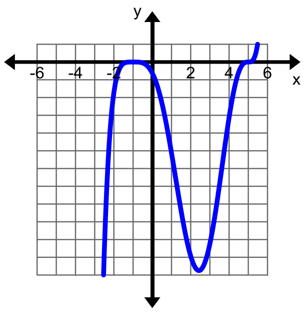
$(-1.5,0)$, $(1,0)$, $(5,0)$

$\lim_{x\to-\infty} f(x) \to -\infty$

$\lim_{x\to\infty} f(x) \to +\infty$

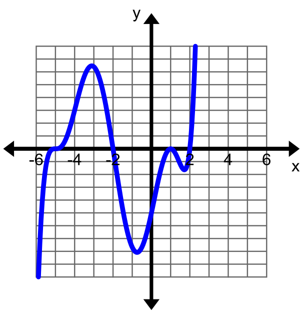
Practice Exercises E (pg. 18)

1.



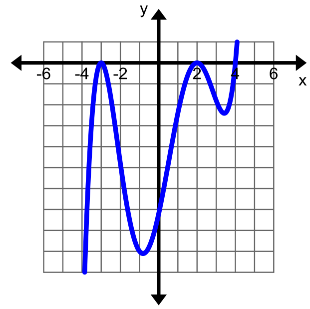
Zero	Multiplicity	Touch/Cross
$(-1,0)$	4	touch
$(5,0)$	5	cross

3.



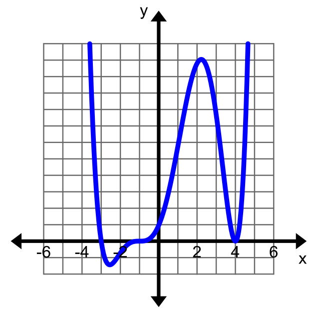
Zero	Multiplicity	Touch/Cross
$(2,0)$	1	Cross
$(-2,0)$	1	Cross
$(-5,0)$	3	Cross
$(1,0)$	2	Touch

5.



Zero	Multiplicity	Touch/Cross
$(2,0)$	2	Touch
$(-3,0)$	2	Touch
$(4,0)$	1	Cross

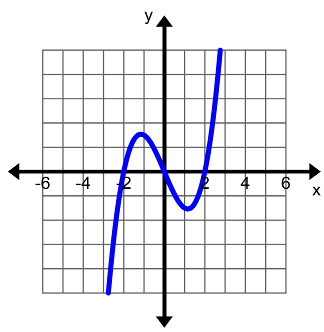
7.

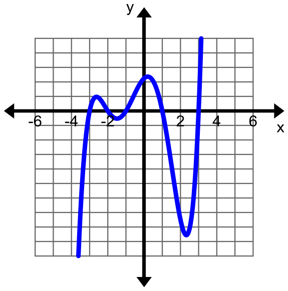


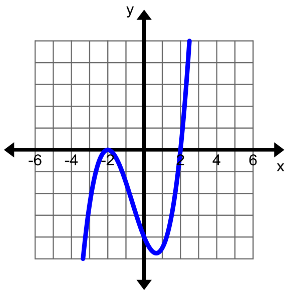
Zero	Multiplicity	Touch/Cross
$(4,0)$	2	Touch
$(-1,0)$	3	Cross
$(-3,0)$	1	Cross

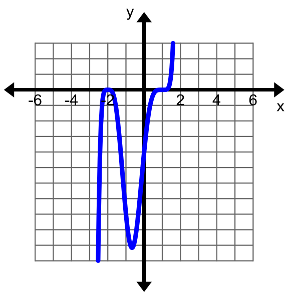
Practice Exercises F (pg. 21)

1.



3. 

5. 

7. 

Unit 2 Cluster 6 (A.APR.4, A.APR.5, and N.CN.8)

Practice Exercises A (pg. 22)

1. $x^2 - x - 20$
3. $64x^3 - 48x^2y + 12xy^2 - y^3$
5. $8x^3 - 60x^2 + 150x - 125$
7. $81x^2 - 64y^2$
9. $x^2 - 16x + 39$
11. $169x^2 + 64$
13. $100x^2 + 16$
15. $x^3 - 1331y^3$
17. $16x^2 - 121$

Practice Exercises B (pg. 24)

1. $(3x - y)(9x^2 + 3xy + y^2)$
3. $(3x - 2)^3$
5. $(3x - 8i)(3x + 8i)$
7. $(x + 11)(x + 8)$
9. $(7x + 2y)(49x^2 + 14xy + 4y^2)$
11. $(x - 5)^3$
13. $(12x - 5i)(12x + 5i)$
15. $(x + 9)(x - 5)$
17. $(x - 3)(x - 6)$
19. $x = \dfrac{5 + \sqrt{37}}{2}, \dfrac{5 - \sqrt{37}}{2}$
21. $x = 2, -1$
23. $x = -\dfrac{1}{3}, -2$

Practice Exercises C (pg. 25)

1. $(x - 2 + i)(x - 2 - i)$
3. $(x + 2 + 2i)(x + 2 - 2i)$
5. $(x + 2 - \sqrt{3}i)(x + 2 - \sqrt{3}i)$
7. $(x + \sqrt{6}i)(x - \sqrt{6}i)$
9. $(x + 3)\left(x - \dfrac{3}{2} + \dfrac{3\sqrt{3}i}{2}\right)\left(x - \dfrac{3}{2} - \dfrac{3\sqrt{3}i}{2}\right)$
11. $(x + 1)(x - 1 + \sqrt{3}i)(x - 1 - \sqrt{3}i)$
13. $(x + 2)(x + 3)(x + 2i)(x - 2i)$
15. $(x - 1 + \sqrt{2}i)(x - 1 - \sqrt{2}i)(x + i)(x - i)$

Practice Exercises D (pg. 28)

1. $(x + 3)^6 = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$
3. $(2x - 1)^5 = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$
5. $(4x - 3y)^5 = 1024x^5 - 3840x^4y + 5760x^3y^2 - 4320x^2y^3 + 1620xy^4 - 243y^5$

Unit 2 Cluster 3 (A.SSE.4 and Honors)

Practice Exercises A (pg. 34)

1. $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}=\frac{1769}{3600}$

3. $1+3+7+15+31=57$

5. $\frac{3}{5}+\frac{4}{6}+\frac{5}{7}+\frac{3}{4}+\frac{7}{9}=\frac{4421}{1260}$

7. $\sum_{k=1}^{7} 2k+3$

9. $\sum_{k=1}^{10} \frac{1}{2k}$

11. $\sum_{k=1}^{9} 40-3k$

Practice Exercises B (pg. 36)

1. $\frac{1\left(1-4^7\right)}{1-4}=5461$

3. $\frac{4\left(1-(-3)^8\right)}{1-(-3)}=-6560$

5. $\frac{1\left(1-\left(\frac{1}{5}\right)^6\right)}{1-\frac{1}{5}}=\frac{3096}{3125}$

7. $\frac{-2\left(1-(-3)^5\right)}{1-(-3)}=-122$

9. $\frac{-2\left(1-6^5\right)}{1-6}=-3110$

11. $\frac{3\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}=\frac{189}{32}$

13. \$15,304,304

15. \$1,260,008

Practice Exercises C (Honors) (pg. 40)

1. $r=10$, diverges

3. $r=\frac{1}{4}$, converges

5. $r=1.02$, diverges

7. $S=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$

9. $S=\frac{2}{1-\left(-\frac{1}{5}\right)}=\frac{5}{3}$

11. $S=\frac{12}{1-\frac{1}{2}}=24$

13. $S=\frac{3}{1-0.1}=\frac{10}{3}$

15. $S=\frac{1}{1-\frac{e}{3}}=\frac{3}{3-e}$

17. $S=\frac{-2}{1-0.6}=-5$

19. $4+2\sum_{k=1}^{\infty}3.92(0.98)^{k-1}=4+2\left(\frac{3.92}{1-0.98}\right)$
396 feet

Practice Exercises D (pg. 43)

1.	816	3.	-285	5.	610
7.	950	9.	-435	11.	301

13. a. $\sum_{k=1}^{25} 2k+20$ b. 1150 c. $\$10,637.50$

15. $\$532,500$

Unit 2 Cluster 7 (A.APR.6, and A.APR.7)

Practice Exercises A (pg. 46−47)

1. $4x^2-2x-1$

3. $3x+4+\frac{9}{x}$

5. $x-5$

7. $6x+1$

9. $\frac{1}{2x-7}$

11. $\frac{x+7}{x-6}$

13. $\frac{x-7}{x+10}$

15. $\frac{3x-2}{x+1}$

17. $\frac{1}{x^2-2x+4}$

19. $\frac{x-4}{x^2-4x+16}$

21. $\frac{1}{2x+1}$

Practice Exercises B (pg. 49)

1. $x-5-\frac{50}{x-5}$

3. $x-5+\frac{1}{x-4}$

5. $4x+2+\frac{5}{x-1}$

7. $2x-1+\frac{1}{3x-2}$

9. $x^2-x+1-\frac{2}{x+1}$

11. $2x^2-x+1-\frac{5}{x+2}$

13. $3x^2-x+4+\frac{10}{2x-3}$

15. $3x^2+x-1-\frac{4}{x-2}$

Practice Exercises C (pg. 52)

1. $x^2+2x-3-\frac{12}{-5x^2+9x-2}$

3. $x^2+3x-\frac{13}{x^2+5x+2}$

5. $2x^2-\frac{5}{4x^2-x-9}$

Practice Exercises D (pg. 55)

1. $\frac{4y}{3x^3}$, rational expression
3. $2x^5$, rational expression
5. $\frac{8x^4}{5}$, rational expression
7. $\frac{2}{x-5}$, rational expression
9. $\frac{2}{3x}$, rational expression
11. $\frac{2}{x}$, rational expression
13. $\frac{(x-2)(3x+2)}{(x+1)(2x+1)}$, rational expression
15. $\frac{x+2}{4x(x+6)}$, rational expression
17. $\frac{5x^3}{x-5}$, rational expression
19. $\frac{(x+2)(x+5)}{3(x-1)}$, rational expression
21. $\frac{x^2+4x+16}{(x+4)(x+4)}$, rational expression
23. $\frac{x(5x-3)}{5(x-4)}$, rational expression
25. $\frac{15}{4x+3}$, rational expression

YOU DECIDE
Rational functions are closed under multiplication and division because all of the answers in Exercises D were rational functions.

Practice Exercises E (pg. 60)

1. $\frac{2}{x}$, rational expression
3. 2, rational expression
5. $\frac{1}{5x+7}$, rational expression
7. $\frac{12-x}{2(x+1)}$, rational expression
9. $\frac{11}{6(x-5)}$, rational expression
11. $\frac{-3}{x-1}$, rational expression
13. $\frac{1}{x+7}$, rational expression
15. $\frac{x^2+14x-49}{x^2-49}$, rational expression
17. $\frac{-x-3}{(x-3)(x-2)}$, rational expression
19. $\frac{5x+5}{(x-2)(x+2)(x+3)}$, rational expression
21. $\frac{40}{(x-5)(x+5)(x+5)}$, rational expression
23. $\frac{13x-4}{(x-2)(2x+3)}$, rational expression
25. $\frac{-2x^2-3x-4}{x(x-1)(x+2)}$, rational expression

YOU DECIDE
Rational functions are closed under addition and subtraction because all of the answers in Exercises E were rational functions.

Unit 2 Cluster 8 (A.REI.2)

Practice Exercises A (pg. 62)

1. $x \neq 9$
3. $x \neq -6$ or $x \neq 1$
5. $x \neq 0$
7. $x \neq 0$ or $x \neq 1$
9. $x \neq -\frac{2}{3}$ or $x \neq 10$

Practice Exercises B (pg. 65)

1. $x = 12$ or $x = -1$
3. $x = -9$
5. $x = -7$
7. $x = -4$
9. $x = \frac{11}{7}$
11. no solution
13. $x = -3$
15. no solution
17. $x = 2$ or $x = 3$
19. no solution
21. $x = -2$ or $x = 6$
23. $x = -\frac{10}{3}$

Practice Exercises C (pg. 69)

1. $x=11$	3. $x=-3$
5. $x=13$	7. $x=25$
9. $x=15$	11. $x=33$
13. $x=20$	15. No solution
17. $x=-1$	19. $x=-125$
21. $x=-5$	23. $x=9$
25. $x=7$ or $x=3$	27. $x=2$ or $x=-1$

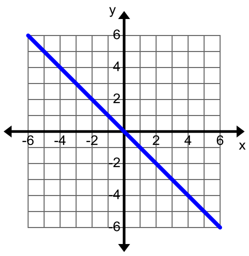
Practice Exercises D (pg. 71)

1. $x=1,\ x=-15$	3. $x=20$
5. $x=1$	7. $x=6$
9. $x=-12,\ x=-28$	

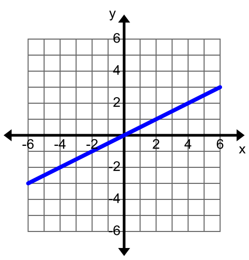
Unit 4 Clusters 3 and 5 (F.IF.7b,e and F.BF.3)

Practice Exercises A (pg. 84)

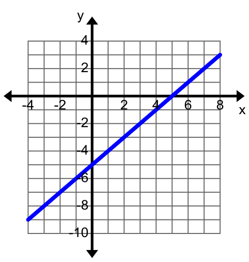
1a.



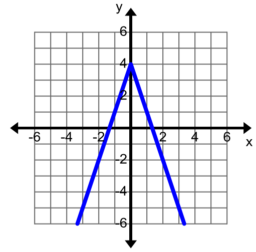
1b.



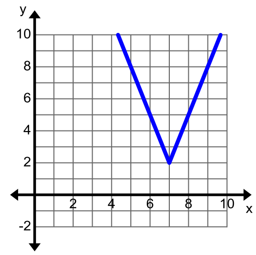
1c.



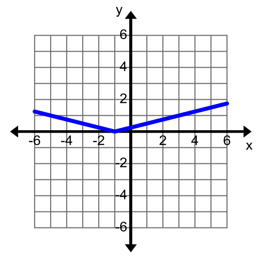
3a.



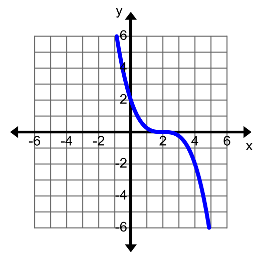
3b.



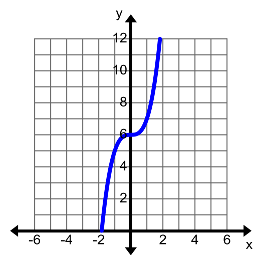
3c.



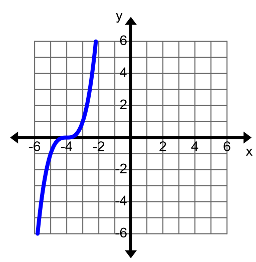
5a.



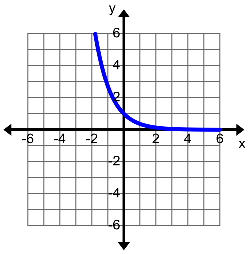
5b.



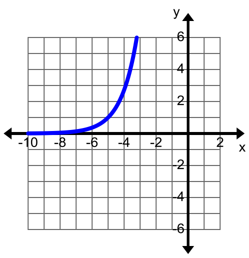
5c.



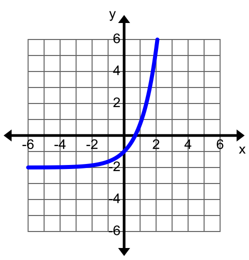
7a.



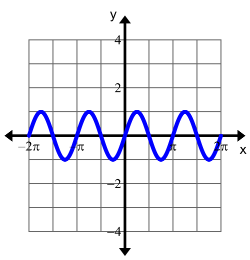
7b.



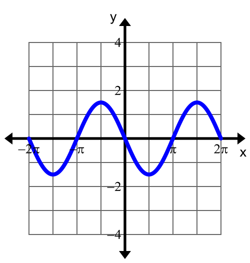
7c.



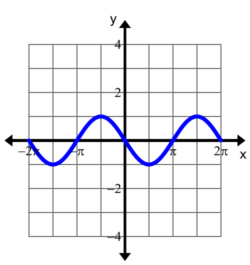
9a.



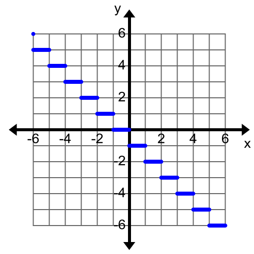
9b.



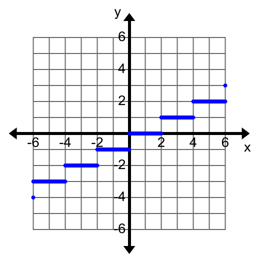
9c.



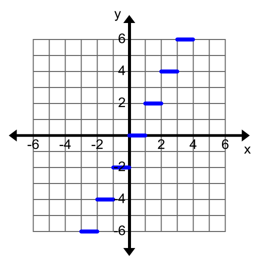
11a.



11b.



11c.

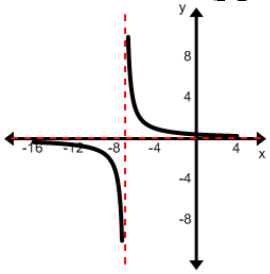


Practice Exercises B (pg. 87)

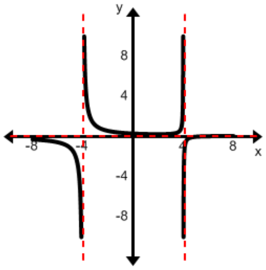
1. yes, translated down 4
3. yes, stretched vertically by a factor of 4, translated down 4 units and left 0.5 units.
5. yes, reflected over the *x*-axis, stretched vertically by a factor of 2
7. yes, translated 3 units left and down 5 units
9. no
11. yes, translated 1 unit left
13. reflected over the *x*-axis, stretched vertically by a factor of 2, translated 3 units left and up 4 units
15. reflected over the *x*-axis, translated 3 units to the right and down 2 units
17. reflected over the *x*-axis, stretched vertically by a factor of 4, translated 2 units to the right

Unit 4 Cluster 3 HONORS (F.IF.7d)

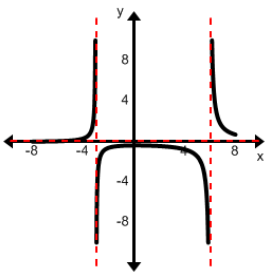
Practice Exercises A (pg. 94)

1.


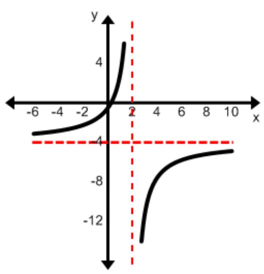
D: $(-\infty,-7)\cup(-7,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

3.


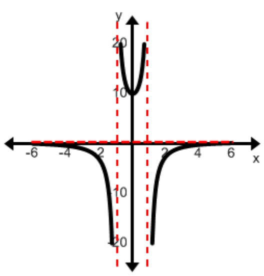
D: $(-\infty,-4)\cup(-4,4)\cup(4,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

5.


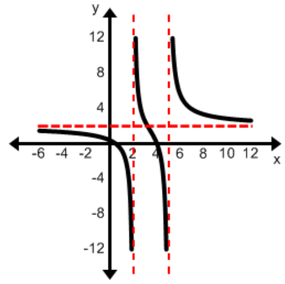
D: $(-\infty,-3)\cup(-3,6)\cup(6,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

7.


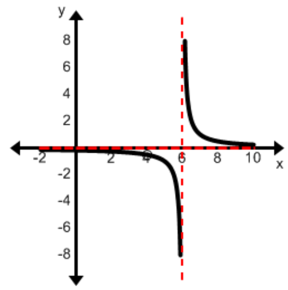
D: $(-\infty,2)\cup(2,\infty)$; R: $(-\infty,-4)\cup(-4,\infty)$

9.


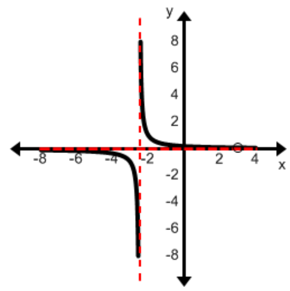
D: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$; R: $(-\infty,0)\cup(10,\infty)$

11.


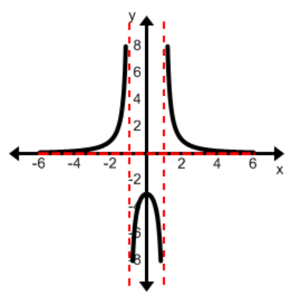
D: $(-\infty,2)\cup(2,5)\cup(5,\infty)$; R: $(-\infty,\infty)$

13.


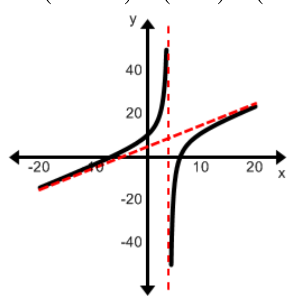
D: $(-\infty,4)\cup(4,6)\cup(6,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

15.


D: $\left(-\infty,-\frac{5}{2}\right)\cup\left(-\frac{5}{2},3\right)\cup(3,\infty)$; R: $(-\infty,0)\cup(0,\infty)$

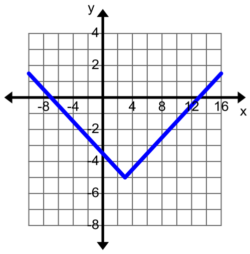
17.


D: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

19.


D: $(-\infty,4)\cup(4,\infty)$

Unit 4 Cluster 2 (F.IF.4, and F.IF.5)
Practice Exercises A (pg. 98)

1a. 

1b. $(-7,0)$, $(13,0)$, and $\left(0,-\frac{3}{5}\right)$

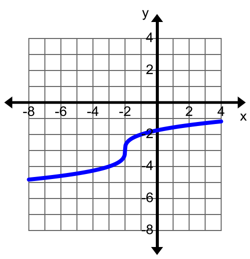
1c. Minimum: $(3,-5)$

1d. Increasing: $(3,\infty)$, decreasing: $(-\infty,3)$

1e. Positive: $(-\infty,-7)\cup(13,\infty)$, negative: $(-7,13)$

1f. $\lim\limits_{x\to\infty} f(x)=\infty$, $\lim\limits_{x\to-\infty} f(x)=\infty$

1g. No symmetry

3a. 

3b. $(25,0)$ and $(0,-1.740)$

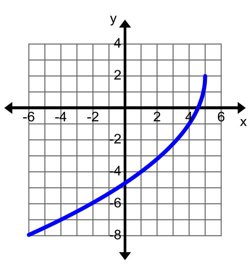
3c. none

3d. Increasing: $(-\infty,\infty)$

3e. Positive: $(25,\infty)$, negative: $(-\infty,25)$

3f. $\lim\limits_{x\to\infty} f(x)=\infty$, $\lim\limits_{x\to-\infty} f(x)=-\infty$

3g. No symmetry

5a. 

5b. $\left(\frac{41}{9},0\right)$ and $(0,-3.708)$

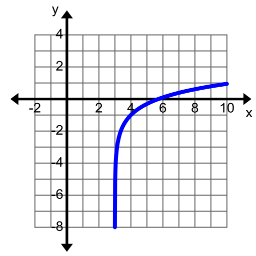
5c. Maximum: $(5,2)$

5d. Increasing: $(-\infty,5)$

5e. Positive: $\left(\frac{41}{9},5\right)$, negative: $\left(-\infty,\frac{41}{9}\right)$

5f. $\lim\limits_{x\to5^-} f(x)=2$, $\lim\limits_{x\to-\infty} f(x)=-\infty$

5g. No symmetry

7a. 

7b. $(6,0)$

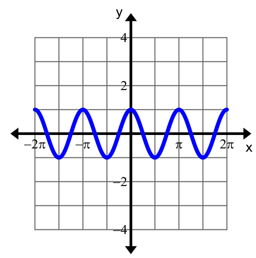
7c. None

7d. Increasing: $(3,\infty)$

7e. Positive: $(6,\infty)$, negative: $(3,6)$

7f. $\lim\limits_{x\to\infty} f(x)=\infty$, $\lim\limits_{x\to3^+} f(x)=-\infty$

7g. No symmetry

9a. 

9b. $\left(\pm\frac{\pi}{4}k,0\right)$ where k is odd, and $(0,1)$

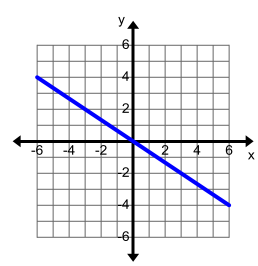
9c. Absoluate maximum of 1 and absolute minimum of -1

9d. alternating increasing and decreasing in periodic waves

9e. alternating positive and negative in periodic waves

9f. no end behavior because the values oscillate between -3 and 3 and approach no limit

9g. Even symmetry

11a. 

11b. $(0,0)$

11c. None

11d. Decreasing: $(-\infty,\infty)$

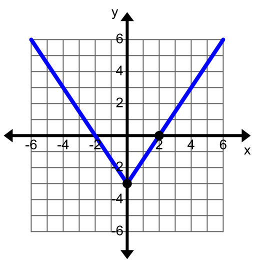
11e. Positive: $(-\infty,0)$, negative: $(0,\infty)$

11f. $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$

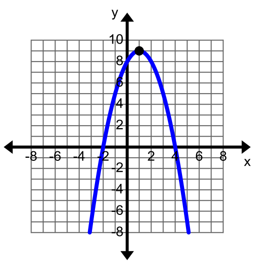
11g. Odd symmetry

Practice Exercises B (pg. 100)

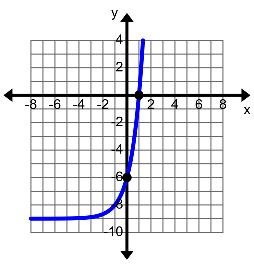
1.



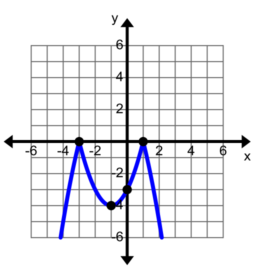
3.



5.



7.



Practice Exercises C (pg. 104−105)

1. $(-\infty,-4)\cup(-4,4)\cup(4,\infty)$

3. $(-\infty,\infty)$

5. $(-\infty,\infty)$

7. $(-\infty,\infty)$

9. $(-\infty,\infty)$

11. $(-\infty,-5)\cup(-5,2)\cup(2,\infty)$

13. $(-\infty,\infty)$

15. $(-\infty,\infty)$

17. $(-\infty,\infty)$

19. $(0,12)$

21. $(0,7.5)$

Unit 4 Cluster 3 (F.IF.9)

Practice Exercises A (pg. 109−112)

1. **Function A**
   a. increasing $(1,\infty)$
   b. positive $(1,\infty)$
   c. minimum $(1,4)$
   d. domain $[1,\infty)$; range $[4,\infty)$
   Function B
   a. decreasing $(1,\infty)$
   b. positive $(1,17)$; negative $(17,\infty)$
   c. maximum $(1,4)$
   d. domain $[1,\infty)$; range $(-\infty,4]$

Both functions have the same domain. Function A is increasing on its entire domain while Function B is decreasing on its entire domain. Function B is a reflection of Function A over the line $y=4$.

3. **Function A**
   a. intercepts $(-6,0),(0,0),(6,0)$
   b. maximum $(-3,4)$; minimum $(3,-4)$
   c. range $[-4,4]$
   d. odd symmetry
   Function B
   a. intercepts $(-6,0),(0,0),(6,0)$
   b. maximum $(3,4)$; minimum $(-3,-4)$
   c. range $[-4,4]$
   d. odd symmetry

Both functions have the same intercepts, the same range, and the same symmetry. The maximums and minimums occur at different places. Function B is a reflection of function A over the *y*-axis or the *x*-axis.

5. **Function A**
   a. intercepts $(-1,0),(3,0),(0,3)$
   b. increasing $(-\infty,1)$; decreasing $(1,\infty)$
   c. maximum $(1,6)$
   d. domain $(-\infty,\infty)$; range $(-\infty,6]$
   Function B
   a. intercepts $(0,0),(2,0)$
   b. increasing $(-\infty,1)$; decreasing $(1,\infty)$
   c. maximum $(1,3)$
   d. domain $(-\infty,\infty)$; range $(-\infty,3]$

Both functions are increasing and decreasing on the same intervals. Both functions have their maximums occur at the same place, but the maximum value of Function A is three more than Function B. Function B is Function A shifted down three units.

7. The rocket that is graphed is in the air longer. The graphed rocket will also have the greatest maximum because it is in the air the longest.

Unit 4 Cluster 2 (F.IF.6)

Practice Exercises A (pg. 115–117)

1.	5	3.	$-a-3$
5.	1	7.	$\frac{1}{5}$
9.	$\frac{2}{5}$	11.	$\frac{3}{5}$
13.	8	15.	5
17.	0.9242	19.	$\frac{8}{3\pi}$
21.	$-\frac{6}{\pi}$	23.	-23

25. the average temperature increased 3.428 $°F$ each month from March to October
27. the percentage of the labor force in unions decreased by 0.53 percent each year from 1975 to 1995.
29. the height of baby boys increased 0.5 inches per month from 6 to 16 months.
31. The liquid decreased in temperature 4.375 degrees Fahrenheit per minute for 4 to 12 minutes.

Unit 2 Cluster 9 (A.REI.11)

Practice Exercises A (pg. 119)
1. at $x=-1$ and $x=2$
3. at $x=-2$ and $x=3$

Practice Exercises B (pg. 121)
1. $x=0.209$ and $x=4.791$
3. $x=-0.679$ and $x=1.179$
5. $x=-0.484$ and $x=4.776$
7. $x=2$, $x=6.382$ and $x=8.618$
9. $x=-1$ and $x=5.372$
11. $x=0.712$ and $x=5.41$
13. $x=14.186$ and $x=-15.048$
15. $x=-10.181$ and $x=-2.146$

Unit 4 Cluster 1 (A.CED.1, A.SSE.2, and A.CED.4)

Practice Exercises A (pg. 125)
1. 3 inches
3. youngest 9, middle 10, oldest 12
5. 2.4 hours
7. 12 hours
9. 82.645 psi

Practice Exercises B (pg. 127)

1.	$x=3$, $x=-7$	3.	$x=\frac{8}{3}$, $x=3$
5.	$x=1$	7.	$x=-\frac{3}{2}$, $x=-3$

9. $x=-\frac{2}{3}$, $x=-\frac{4}{3}$
11. $x=-\frac{1}{8}$, $x=27$
13. $x=-\frac{1}{3}\sqrt{\frac{1}{3}}$, $x=\frac{1}{3}\sqrt{\frac{1}{3}}$
15. $x=0$, $x=1$
17. $x=4$, $x=0$, $x=-4$
19. $x=0$, $x=6$, $x=-3$
21. $x=3$, $x=1$, $x=-1$

Practice Exercises C (pg. 130)

1. $(-\infty,-4]\cup[3,\infty)$
3. $(-\infty,-4]\cup[1,\infty)$
5. $(-\infty,-5)\cup(0,3)$
7. $(-\infty,-1]\cup[0,7]$
9. $(-\infty,-3)\cup[0,\infty)$
11. $(-\infty,-3)\cup[-2,3)$
13. $(0,2)\cup(2,\infty)$
15. $\left[\frac{4}{3},\infty\right)$
17. $(0,1.343)$ seconds
19. $(0,1.5]\cup[2.628,7)$ inches
21. $(2.863,9.209)$ centimeters

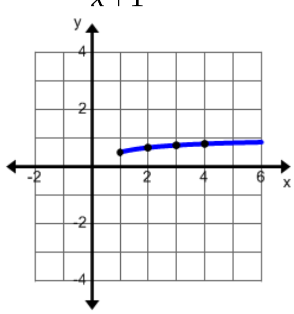
Practice Exercises D (pg. 132)

1. $B=\pm\sqrt{\dfrac{2Vm}{qr^2}}$
3. $c=\dfrac{(2ax+b)-b^2}{-4a}$
5. $y=\pm\sqrt{b^2-\dfrac{b^2x^2}{a^2}}$
7. $b=\pm\sqrt{k^2+4ac}$
9. $a_n=\dfrac{2S}{n}-a_1$
11. $r=\sqrt[3]{\dfrac{3V}{4\pi}}$
13. $r_2=\dfrac{Rr_1}{r_1-R}$
15. $y_2=y_1\pm\sqrt{d^2-(x_2-x_1)^2}$

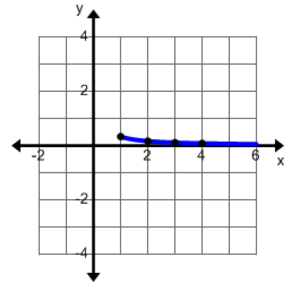
Unit 4 Cluster 1 (A.CED.2 and A.CED.3)

Practice Exercises A (pg. 136-138)

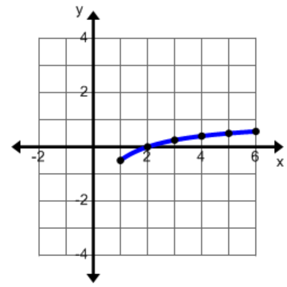
1. $f(x)=\dfrac{x}{x+1}$ for $x\geq 1$



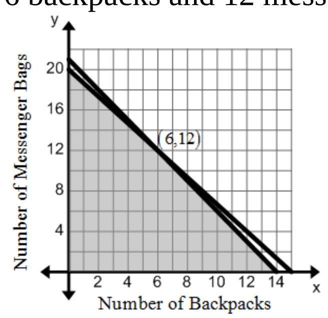
3. $f(x)=\dfrac{1}{3x}$ for $x\geq 1$



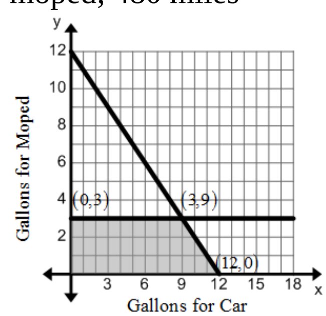
5. $f(x)=\dfrac{x-2}{x+1}$ for $x\geq 1$



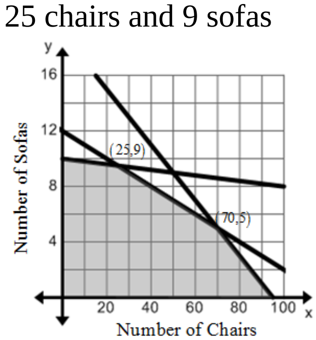
7. 6 backpacks and 12 messenger bags



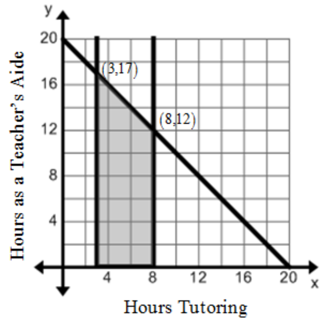
9. 9 gallons for the car, 3 gallons for the moped; 480 miles



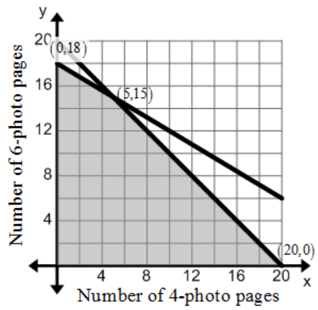
11. 25 chairs and 9 sofas



13. 8 hours tutoring, 12 hours as a teacher's aide.



15. 5 of 4 photo pages, 15 of 6 photo pages; 110 photos



Unit 4 Cluster 4 (F.BF.1, F.BF.1c honors)

Practice Exercises A (pg. 145)

1a. $h(x)=\sqrt{x-4}+2-3x^2$; $[4,\infty)$

1b. $h(x)=\sqrt{x-4}+2+3x^2$; $[4,\infty)$

1c. $h(x)=-3x^2\sqrt{x-4}-6x^2$; $[4,\infty)$

1d. $h(x)=\dfrac{\sqrt{x-4}}{-3x^2}$; $[4,\infty)$

3a. $h(x)=\sin x+x^3-3$; $(-\infty,\infty)$

3b. $h(x)=\sin x-x^3+3$; $(-\infty,\infty)$

3c. $h(x)=x^3\sin x-3\sin x$; $(-\infty,\infty)$

3d. $h(x)=\dfrac{\sin x}{x^3-3}$; $(-\infty,\sqrt[3]{3})\cup(\sqrt[3]{3},\infty)$

5a. $h(x)=x^3-7x+6$; $(-\infty,\infty)$

5b. $h(x)=x^3-2x^2+7x-6$; $(-\infty,\infty)$

5c. $h(x)=x^5-8x^4+13x^3-6x^2$; $(-\infty,\infty)$

5d. $h(x)=\dfrac{x^2}{x-6}$; $(-\infty,1)\cup(1,6)\cup(6,\infty)$

7a. $h(x)=\cos(3x)+\sqrt[3]{x+1}$; $(-\infty,\infty)$

7b. $h(x)=\cos(3x)-\sqrt[3]{x+1}$; $(-\infty,\infty)$

7c. $h(x)=\cos(3x)\sqrt[3]{x+1}$; $(-\infty,\infty)$

7d. $h(x)=\dfrac{\cos(3x)}{\sqrt[3]{x+1}}$; $(-\infty,-1)\cup(-1,\infty)$

Practice Exercises B (pg. 147)

1. -5
3. 4
5. 4
7. 0
9. 25
11. 3
13. $P(x)=0.75x^2+50x-19{,}900$; \$750,049,980,100
15. $\overline{C}(x)=0.5x^2-34x+1213$; \$2813

Practice Exercises C (pg. 151)

1a. $h(x)=\dfrac{-8x+57}{x-7}$; $(-\infty,7)\cup(7,\infty)$

1b. $h(x)=\dfrac{1}{x-15}$; $(-\infty,15)\cup(15,\infty)$

1c. $h(x)=x-16$; $(-\infty,\infty)$

1d. $h(x)=\dfrac{-x+7}{7x-50}$; $(-\infty,7)\cup(7,\frac{50}{7})\cup(\frac{50}{7},\infty)$

3a. $h(x)=\sqrt{x^2-9}$; $(-\infty,-3)\cup(3,\infty)$

3b $h(x)=x-9$; $[6,\infty)$

3c. $h(x)=\sqrt{\sqrt{x-6}-6}$; $[42,\infty)$

3d. $h(x)=x^4-6x^2+6$; $(-\infty,\infty)$

5a. $h(x)=x$; $(-\infty,\infty)$

5b. $h(x)=x$; $(-\infty,\infty)$

5c. $h(x)=\sqrt[3]{\sqrt[3]{x-2}-2}$; $(-\infty,\infty)$

5d. $h(x)=x^9+6x^6+12x^3+10$; $(-\infty,\infty)$

7a. $h(x)=\cos(4-x)$; $(-\infty,\infty)$

7b. $h(x)=4-\cos(x)$; $(-\infty,\infty)$

7c. $h(x)=\cos(\cos(x))$; $(-\infty,\infty)$

7d. $h(x)=x$; $(-\infty,\infty)$

Practice Exercises D (pg. 156)

1. 9
3. 3
5. 1
7. 19
9. $\frac{7}{13}$
11. It is the same amount whether you apply the coupon or the tax first.

Unit 4 Cluster 5 (F.BF.4, F. BF.4b,c,d honors)

Practice Exercises A (pg. 159)

1. $f^{-1}(x)=\dfrac{x-8}{-6}$
3. $f^{-1}(x)=\dfrac{x-5}{3}$
5. $f^{-1}(x)=2+\sqrt{x+16}$
7. $f^{-1}(x)=x^2-4,\ x\ge 0$
9. $f^{-1}(x)=-\dfrac{1}{4}x^2+3,\ x\ge 0$
11. $f^{-1}(x)=\dfrac{x+5}{x-3}$
13. $f^{-1}(x)=\dfrac{-2x-6}{3x-7}$
15. $f^{-1}(x)=\sqrt[3]{2x+6}$
17. $f^{-1}(x)=\sqrt[3]{x-5}+2$
19. $f^{-1}(x)=\left(\dfrac{x-7}{-2}\right)^3+5$
21. $f^{-1}(x)=(x+4)^3+1$

Practice Exercises B (pg. 160)

1. $(f\circ g)=x,\ (g\circ f)=x$
3. $(f\circ g)=x,\ (g\circ f)=x$
5. $(f\circ g)=x,\ (g\circ f)=x$
7. $(f\circ g)=x,\ (g\circ f)=x$
9. $(f\circ g)=x,\ (g\circ f)=x$

Practice Exercises C (pg. 162)

1.

x	$f^{-1}(x)$
0.5	-2
1.5	-1
4.5	0
13.5	1
2	40.5

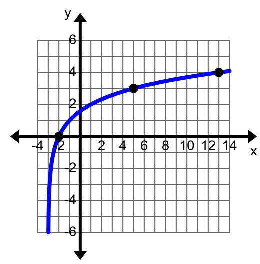
3.

x	$f^{-1}(x)$
1	5
3	6
4	9
5	14
6	21

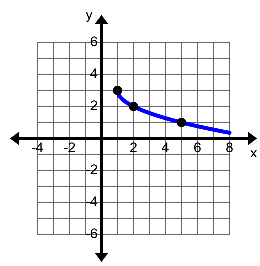
5.

x	$f^{-1}(x)$
1.7	-17
1.6	-12
1.5	-9
1.4	-7
1	-3

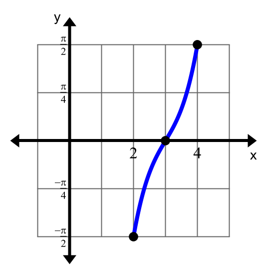
7.



9.



11.



Practice Exercises D (pg. 163)

1. $x \ge 0$ or $x \le 0$
3. $x \ge -5$ or $x \le -5$
5. $x \ge -6$ or $x \le -6$
7. $x \ge 3$ or $x \le 3$
9. $x \ge 2$ or $x \le 2$
11. $x \ge 7$ or $x \le 7$

Unit 4 Cluster 6 (F.LE.4 and F.BF.5)

Practice Exercises A (pg. 167)

1. $4^0 = 1$
3. $9^{-2} = \frac{1}{81}$
5. $7^{-3} = \frac{1}{343}$
7. $\log_{10} \frac{1}{1000} = -3$
9. $\log_6 \frac{1}{6} = -1$
11. $\log_7 7 = 1$
13. 1
15. 0
17. $3 - x$
19. 1
21. $10x + 5$

Practice Exercises B (pg. 169)

1. 343
3. 10
5. 1
7. 1
9. 5
11. 2

Practice Exercises C (pg. 171)

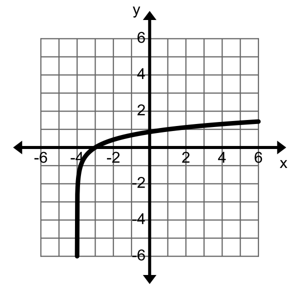
1. $5\log_4 x + 7\log_4 y$
3. $2\log a + 3\log b - 4\log c$
5. $1 + \frac{1}{2}\log_8 3 + \frac{5}{2}\log_8 a$
7. $3 + \log_3 (x-3) - 2\log_3 x - 5\log_3 y$
9. $3\ln x + \frac{1}{2}\ln\left(x^2+1\right) - 4\ln(x+1)$
11. $\log_2\left(\frac{96}{3}\right) = 5$
13. $\ln\left(\frac{x^4 y^7}{z^3}\right)$
15. $\log_3 \sqrt{\frac{xz}{y^3}}$
17. $\ln \frac{x-2}{\left(x^2-4\right)x^3} = \ln \frac{1}{x^3(x+2)}$

Practice Exercises D (pg. 172)

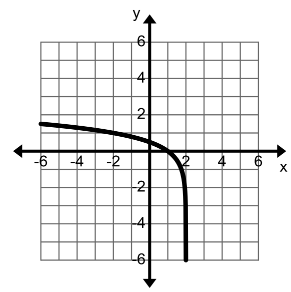
1. -0.693
3. 4.025
5. -0.903
7. 5.000
9. 2.262
11. 1.745

Practice Exercises E (pg. 174)

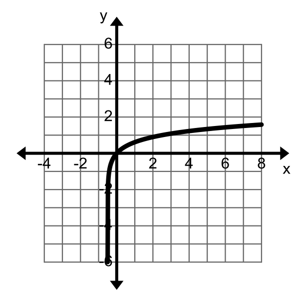
1. $(-4, \infty)$



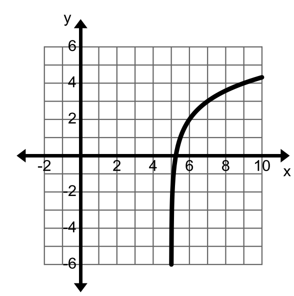
3. $(-\infty, 2)$



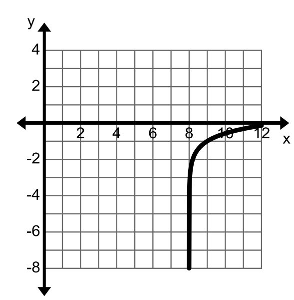
5. $\left(-\frac{1}{2}, \infty\right)$



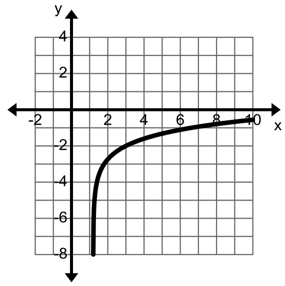
7. $(5, \infty)$



9. $(8, \infty)$



11. $\left(\frac{6}{5},\infty\right)$



Practice Exercises F (pg. 176)

1. $x=5$
3. $x=12$
5. $x=6$
7. $x=12$
9. $x=-3$
11. $x=-\log_2 1.5$ ≈ -0.585
13. $x=\dfrac{\ln 8}{0.6}\approx 3.466$
15. $x=\dfrac{\ln 14}{2\ln 4}+0.5$ $x\approx 1.452$
17. $x=\dfrac{\ln 13}{4\ln 5}+1.75$ $x\approx 2.148$

Practice Exercises G (pg. 178)

1. $f^{-1}(x)=10^{x+2}-7$
3. $f^{-1}(x)=8-e^{\frac{x-5}{2}}$
5. $f^{-1}(x)=3\cdot 2^{x-7}-6$
7. $f^{-1}(x)=\log_5(x-2)+3$
9. $f^{-1}(x)=\log_7(x+3)+2$
11. $f^{-1}(x)=-\frac{1}{2}\log_3\left(-\frac{1}{2}x+\frac{1}{2}\right)+\frac{5}{2}$

Practice Exercises H (pg. 179)

1. $x=\ln 3\approx 1.099$
3. $x=\log_3 1=\dfrac{\ln 1}{\ln 3}=0$
5. $x=\ln 2\approx 0.693,\ x=\ln 1=0$
7. $x=\log_7 6=\dfrac{\ln 6}{\ln 7}\approx 0.921$
9. $x=\log_5 2=\dfrac{\ln 2}{\ln 5}\approx 0.431$

Unit 3 Cluster 1 (G.SRT.9)

Practice Exercises A (Pg. 183)

1. 222.332 ft^2
3. 5.290 cm^2
5. 17.973 ft^2
7. 128.079 m^2
9. 204.985 ft

Unit 3 Cluster 1 (G.SRT.10 and G.SRT.11)

Practice Exercises A (Pg. 187–188)

1. $C=110°,\ a\approx 12.856,\ c\approx 18.794$
3. $C=75°,\ a\approx 4.532,\ c\approx 5.054$
5. $A=77°,\ b\approx 12.686,\ c\approx 9.426$
7. $B\approx 28.822°,\ C\approx 111.178°,\ c\approx 29.013$
9. $A\approx 99.290°,\ C\approx 30.710°,\ a\approx 38.649$
11. $B\approx 41.328°,\ C\approx 89.672°,\ c\approx 42.400$
13. $B\approx 45.805°,\ C\approx 54.196°,\ b\approx 12.376$
15. $A\approx 38.647°,\ B\approx 105.353°,\ b\approx 26.249$ or $A\approx 141.353°,\ B\approx 2.647°,\ b\approx 1.257$
17. no triangle can be formed
19. $A\approx 94.867°, B\approx 47.133°,\ a\approx 33.987$ or $A\approx 9.133°, B\approx 132.867°,\ a\approx 5.414$
21. $B\approx 64.534°, C\approx 52.466°,\ b\approx 10.133$

Practice Exercises B (Pg. 191)

1. $A\approx 26.540°, B\approx 126.460°,\ c\approx 5.080$
3. $A\approx 71.992°, C\approx 68.008°,\ b\approx 54.072$
5. $A\approx 45.530°, B\approx 92.470°,\ c\approx 4.688$
7. $A\approx 22.332°, B\approx 108.21°,\ C\approx 49.458°$
9. $A\approx 27.660°, B\approx 40.536°,\ C\approx 11.804°$
11. $A\approx 34.960°, B\approx 46.826°,\ C\approx 98.213°$
13. Law of Sines, $B\approx 77.109°$ or $B\approx 102.891°$
15. Law of Cosines, $a\approx 5.962$
17. Law of Sines, $c\approx 1.051$
19. Law of Cosines, $C\approx 39.974°$

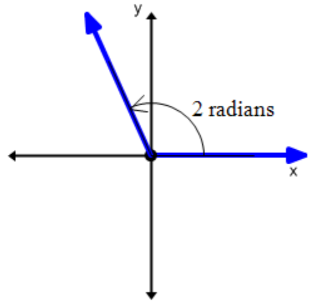
Practice Exercises C (Pg. 194–195)

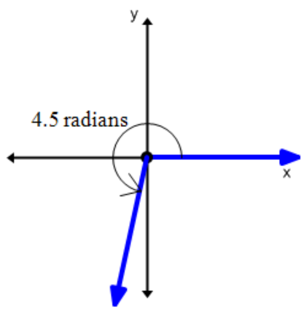
1. 13 miles or 2 miles
3. 4.8 miles
5. distance from A 10.1 miles, distance from B 13.6 miles
7. 93.2°
9. 28.9 feet
11. 87°, it is leaning

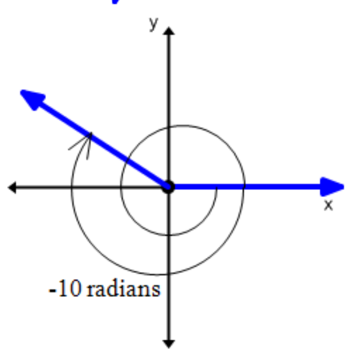
13. 100.2°
15. 214.4 yards

Unit 3 Cluster 2 (F.TF.1, F.TF.2, and F.TF.3)

Practice Exercises A (Pg. 198)

1.


3.


5.


Practice Exercises B (Pg. 206)

1. $0, 2\pi$
3. $\frac{\pi}{4}$
5. $\frac{\pi}{2}$
7. $\frac{3\pi}{4}$
9. π
11. $\frac{5\pi}{4}$
13. $\frac{3\pi}{2}$
15. $\frac{7\pi}{4}$

Practice Exercises C (Pg. 208)

1. $-\frac{\sqrt{2}}{2}$
3. $-\frac{\sqrt{3}}{2}$
5. $\sqrt{3}$
7. -1
9. $-\frac{1}{2}$
11. 1
13. $-\frac{1}{2}$
15. $-\frac{\sqrt{3}}{3}$
17. -1
19. $-\frac{\sqrt{2}}{2}$
21. undefined
23. $-\frac{1}{2}$
25. j
27. c
29. a
31. f
33. g
35. p
37. neither
39. negative
41. positive
43. positive
45. negative
47. negative

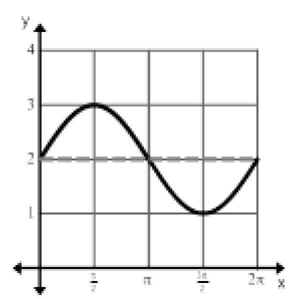
Practice Exercises D (Honors) (Pg. 210)

1. $-\sqrt{3}$
3. $-\frac{2}{\sqrt{3}}$
5. -2
7. $\sqrt{3}$
9. $-\frac{2}{\sqrt{3}}$
11. $\frac{2}{\sqrt{3}}$

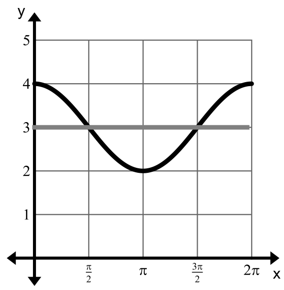
Unit 3 Clusters 2 & 3 (F.TF.2 and F.TF.5)

Practice Exercises A (Pg. 215–216)

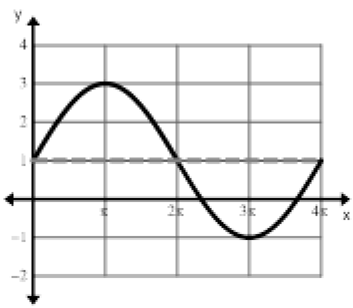
1. Amplitude: 1
   Period: 2π



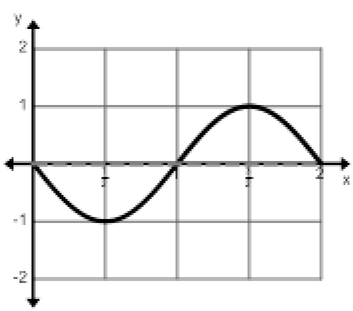
3. Amplitude: 1
Period: 2π



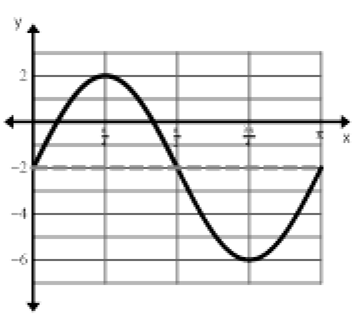
5. Amplitude: 2
Period: 4π



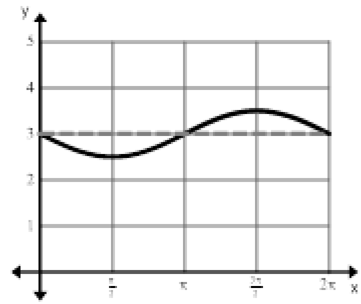
7. Amplitude: 1
Period: 2



9. Amplitude: 4
Period: π



11. Amplitude: $\frac{1}{2}$
Period: 2π



13. Amplitude: 2
Period: π
Midline: $y = -1$
$f(x) = -2\sin(2x) - 1$

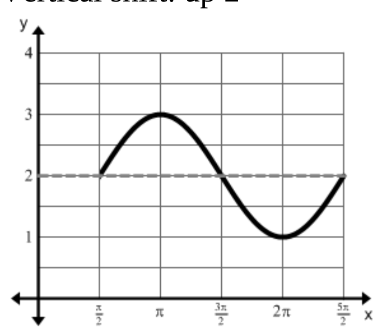
15. Amplitude: 1
Period: 6π
Midline: $y = 3$
$f(x) = \sin\left(\frac{1}{3}x\right) + 3$

17. Amplitude: 4
Period: 8
Midline: $y = 1$
$f(x) = 4\cos\left(\frac{\pi}{4}x\right) + 1$

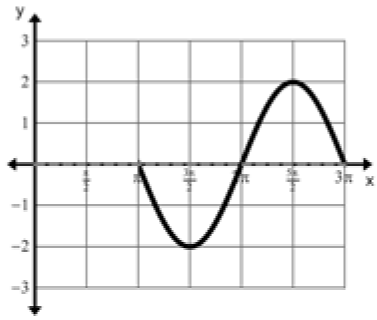
19. Amplitude: 2
Period: 2
Midline: $y = -3$
$f(x) = -2\cos(\pi x) - 3$

Practice Exercises B (Pg. 218)

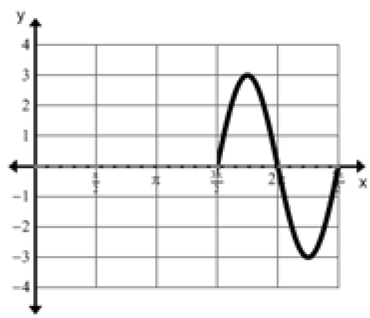
1. Amplitude: 1
Period: 2π
Phase shift: right $\dfrac{\pi}{2}$
Vertical shift: up 2



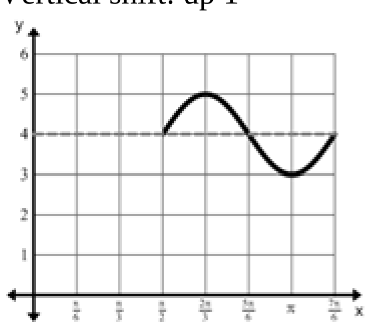
3. Amplitude: 2
Period: 2π
Phase shift: right π
Vertical shift: none



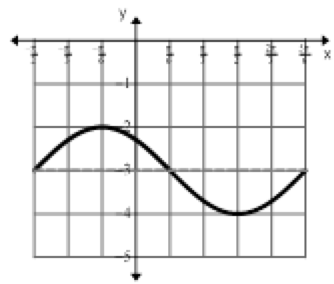
5. Amplitude: 3
Period: π
Phase shift: right $\dfrac{3\pi}{2}$
Vertical shift: none



7. Amplitude: 1
Period: $\dfrac{2\pi}{3}$
Phase shift: right $\dfrac{3\pi}{2}$
Vertical shift: up 1



9. Amplitude: 1
Period: $\dfrac{4\pi}{3}$
Phase shift: left $\dfrac{\pi}{2}$
Vertical shift: down 3



Practice Exercises C (Pg. 220)

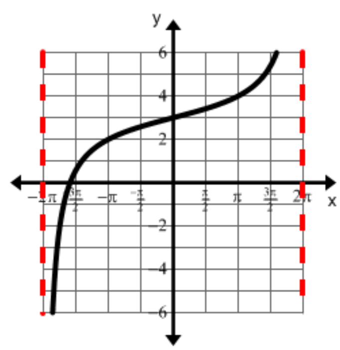
1. $f(t) = 1.8\cos\left(\dfrac{\pi}{8}t\right)$

3. $f(t) = -32\cos\left(\dfrac{\pi}{6}t\right)$

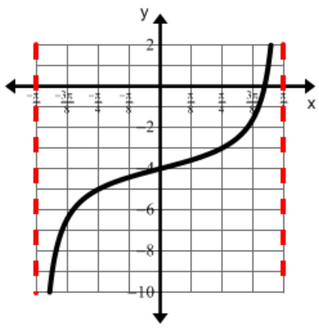
5. $f(x) = \sin(40{,}000\pi x)$

Practice Exercises D (Honors) (Pg. 223)

1. Period: 2π
Asymptotes: $x = \pm 2\pi$
y-intercept: $(0,3)$



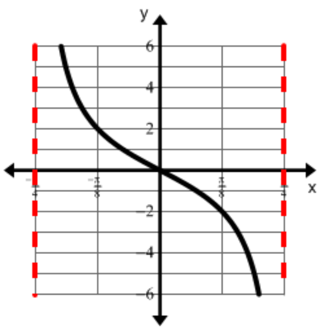
3. Period: π
Asymptotes: $x = \pm\dfrac{\pi}{2}$
y-intercept: $(0,-4)$



5. Period: $\frac{\pi}{2}$

Asymptotes: $x = \pm\frac{\pi}{4}$

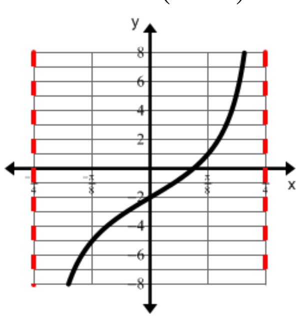
y-intercept: $(0,0)$



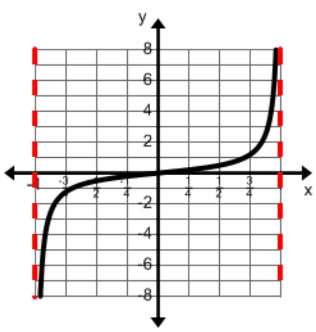
7. Period: $\frac{\pi}{2}$

Asymptotes: $x = \pm\frac{\pi}{4}$

y-intercept: $(0,-2)$



9. Period: 2
Asymptotes: $x = \pm 1$
y-intercept: $(0,0)$



Unit 3 Clusters 2 and 3 Honors (F.TF.4)

Practice Exercises A (Pg. 226)

1. $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$
3. $\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$
5. $\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}$, $\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
7. $\sin\left(\frac{13\pi}{6}\right) = \frac{1}{2}$
9. $\sin\left(\frac{13\pi}{2}\right) = 1$
11. $\sin\left(\frac{61\pi}{3}\right) = \frac{\sqrt{3}}{2}$
13. $\sin\left(\frac{24\pi}{4}\right) = \frac{\sqrt{2}}{2}$
15. $\sin\left(\frac{7\pi}{2}\right) = 1$

Unit 3 Cluster 3 Honors (F.TF.6 and F.TF.7)

Practice Exercises A (Pg. 231)

1. $\frac{\pi}{6}$
3. $\frac{\pi}{2}$
5. $-\frac{\pi}{4}$
7. $-\frac{\pi}{2}$
9. $\frac{\pi}{6}$
11. π
13. $\frac{2\pi}{3}$
15. $\frac{\pi}{3}$
17. 0
19. $-\frac{\pi}{6}$

Practice Exercises B (Pg. 235)

1. 0.8
3. 7
5. $\frac{\sqrt{2}}{2}$
7. $\frac{\pi}{6}$
9. $-\frac{\pi}{2}$
11. $-\frac{\pi}{4}$
13. $-\frac{\pi}{3}$
15. $\frac{\pi}{6}$

17. $\frac{\sqrt{2}}{2}$

19. $\frac{1}{2}$

21. $\frac{24}{25}$

23. $-\frac{4}{3}$

25. $\frac{15}{17}$

27. $\frac{3}{\sqrt{7}}$

29. $\frac{x}{\sqrt{1+x^2}}$

31. $\frac{1}{\sqrt{x^2-1}}$

33. $\frac{x}{\sqrt{x^2+4}}$

35. $\frac{3}{x}$

Unit 3 Cluster 3 Honors Solving Trigonometric Equations

Practice Exercises A (Pg. 238)

1. $x = 0$ or π

3. $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

5. $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

7. $x = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

9. $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$

11. $x = \pi$

13. $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

15. $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

17. $x = \frac{3\pi}{2}$

19. 0.366 or 2.805
21. 1.016 or 4.158
23. 0.406 or 3.548
25. 1.966 or 5.107
27. 1.400 or 4.883

Practice Exercises B (Pg. 241)

1. $x = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

3. $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

5. $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

7. $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

9. $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

11. $x = 0, \frac{\pi}{4}, \pi$ or $\frac{5\pi}{4}$

13. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

15. $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

17. $x = 0, \frac{\pi}{4}, \pi,$ or $\frac{7\pi}{4}$

19. $x = 0, \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$

21. $x = \frac{7\pi}{6}, \frac{3\pi}{2}$ or $\frac{11\pi}{6}$

23. $x = 0$ or π

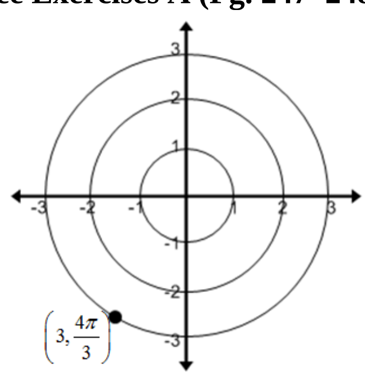
25. 0.0095 seconds

27. 4.111 hours and 8.222 hours

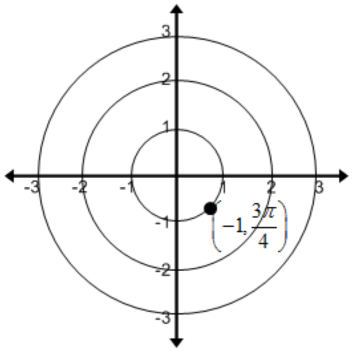
Honors Defining and Using Polar Coordinates

Practice Exercises A (Pg. 247–248)

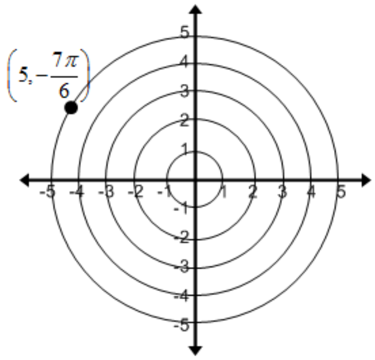
1.



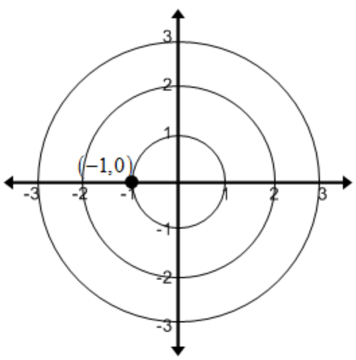
3.



5.



7.



9.
a. $\left(5, \frac{13\pi}{6}\right)$
b. $\left(-5, \frac{7\pi}{6}\right)$
c. $\left(5, -\frac{11\pi}{6}\right)$

11.
a. $\left(10, \frac{11\pi}{4}\right)$
b. $\left(-10, \frac{7\pi}{4}\right)$
c. $\left(10, -\frac{5\pi}{4}\right)$

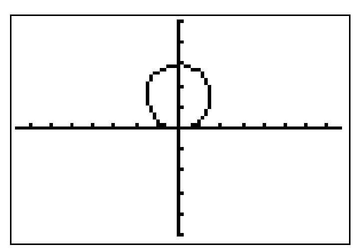
13.
a. $\left(12, \frac{13\pi}{4}\right)$
b. $\left(-12, \frac{\pi}{4}\right)$
c. $\left(12, -\frac{3\pi}{4}\right)$

15.
a. $\left(3, \frac{5\pi}{2}\right)$
b. $\left(-3, \frac{3\pi}{2}\right)$
c. $\left(3, -\frac{3\pi}{2}\right)$

17. D
19. A
21. B
23. C
25. $(0, 4)$
27. $\left(1, \sqrt{3}\right)$
29. $\left(3\sqrt{2}, -3\sqrt{2}\right)$
31. $\left(-\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$
33. $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$
35. $\left(4, \frac{5\pi}{3}\right)$
37. $\left(2, \frac{7\pi}{6}\right)$
39. $(3, \pi)$

Practice Exercises B (Pg. 252)

1. $r = \frac{-3}{\cos\theta}$
3. $r = \frac{4}{\sin\theta}$
5. $r = \frac{5}{2\cos\theta - 3\sin\theta}$
7. $r = \frac{8}{\cos\theta + 5\sin\theta}$
9. $r^2 = 25$
11. $r = 6\cos\theta$
13. $r = \frac{3\sin\theta}{\cos^2\theta}$
15. $r = \frac{4\sin\theta - 8\cos\theta}{\cos^2\theta}$
17. $r = -6\cos\theta - 6\sin\theta$
19.



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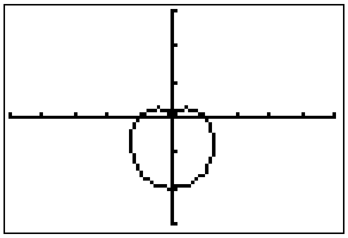

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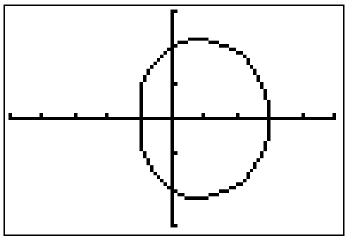
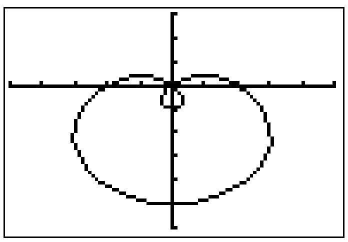
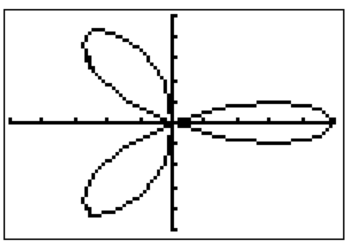


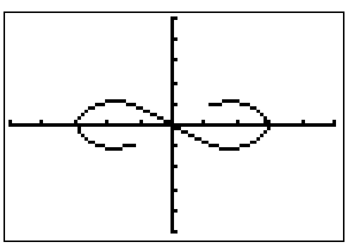
25.



27.



29.



Honors Unit Complex Numbers in Polar Form

Practice Exercises A (Pg. 257–258)

1. $3\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

3. $2\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

5. $5\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

7. $4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

9. $6\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

11. $\sqrt{13}\left(\cos 4.124+i\sin 4.124\right)$

13. $2\sqrt{3}+2i$

15. $-4-4\sqrt{3}i$

17. $2\sqrt{2}-2\sqrt{2}i$

19. $7i$

21. -3

23. $\frac{\sqrt{3}}{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$; $-0.837+0.224i$

25. $2\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$; $0.518-1.932i$

27. $3\left(\cos\frac{41\pi}{30}+i\sin\frac{41\pi}{30}\right)$; $-1.220-2.741i$

29. $\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)$; $0.696+0.123i$

31. $k=0;\ \sqrt[3]{2}\left(\cos\frac{11\pi}{18}+i\sin\frac{11\pi}{18}\right)$

$k=1;\ \sqrt[3]{2}\left(\cos\frac{23\pi}{18}+i\sin\frac{23\pi}{18}\right)$

$k=2;\ \sqrt[3]{2}\left(\cos\frac{35\pi}{18}+i\sin\frac{35\pi}{18}\right)$

33. $k=0;\ 2\sqrt[3]{4}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

$k=1;\ 2\sqrt[3]{4}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$

$k=2;\ 2\sqrt[3]{4}\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)$

35. $k=0;\ \sqrt[4]{108}\left(\cos\frac{\pi}{24}+i\sin\frac{\pi}{24}\right)$

$k=1;\ \sqrt[4]{108}\left(\cos\frac{13\pi}{24}+i\sin\frac{13\pi}{24}\right)$

$k=2;\ \sqrt[4]{108}\left(\cos\frac{25\pi}{24}+i\sin\frac{25\pi}{24}\right)$

$k=3;\ \sqrt[4]{108}\left(\cos\frac{37\pi}{24}+i\sin\frac{37\pi}{24}\right)$

37. $k=0;\ 2\left(\cos 0+i\sin 0\right);\ 2$

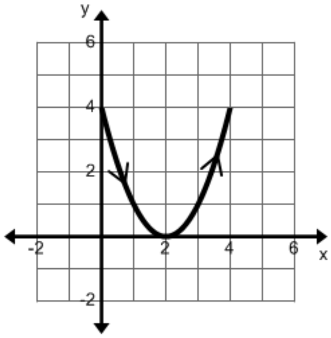
$k=1;\ 2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right);\ -1+\sqrt{3}i$

$k=2;\ 2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right);\ -1-\sqrt{3}i$

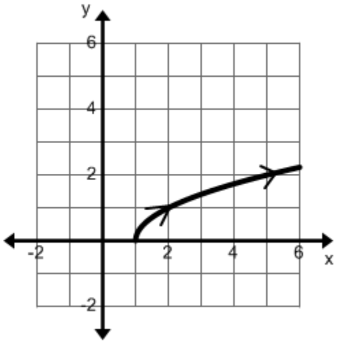
Honors Unit Parametric Equations

Practice Exercises A (Pg. 263–264)

1.



3.



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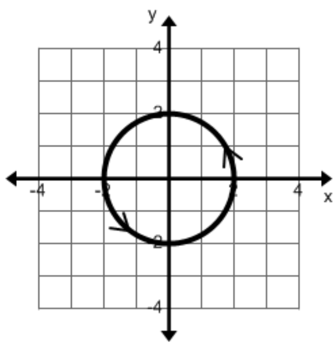
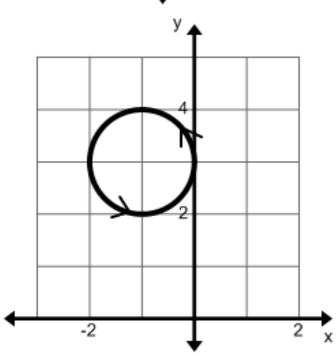
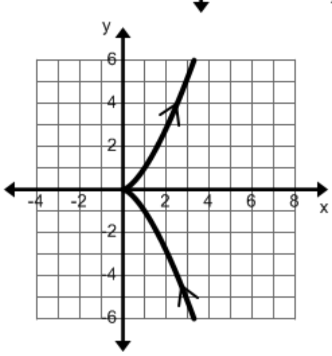


7.



9.



11. $y = x - 1$

13. $y = x^2 - 2x + 3$

15. $y = (x+4)^2$

17. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

19. $\dfrac{(x-1)^2}{9} + \dfrac{(y+1)^2}{4} = 1$; $1 \le x \le 4$

21. $x = 270\cos(48)t$

$y = 270\sin(48)t - 16t^2 + 4$

The arrow hits the ground after about 12.560 seconds.
The arrow lands about 2269.241 feet from where it starts.

23. $x = 65\cos(36)t$

$y = 205\sin(35)t - 16t^2 + 0.5$

The ball will make it over the goal post.

25. $x = 150\cos(35)t$

$y = 150\sin(35)t - 16t^2 + 3$

The ball is in flight for approximately 5.412 seconds.
The ball travel about 664.978 feet horizontally.

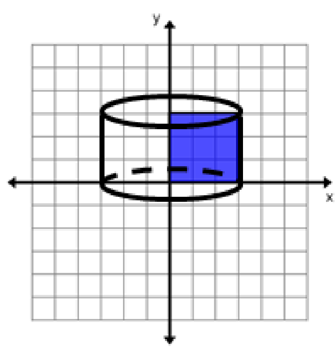
Unit 4 Cluster 7 (G.GMD.4) Two and Three Dimensional Objects

Practice Exercises A (Pg. 268)

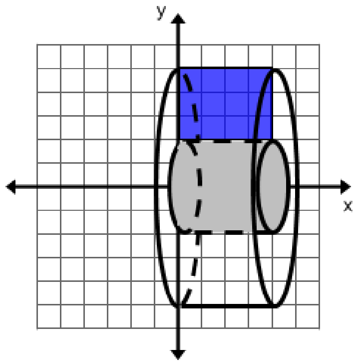
1. circle
3. ellipse
5. triangle
7. pentagon
9. rectangle
11. pentagon
13. circle
15. hexagon
17. heptagon

Practice Exercises B (Pg. 270–271)

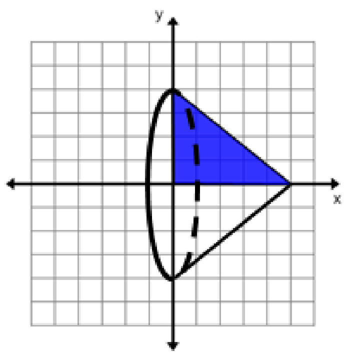
1.



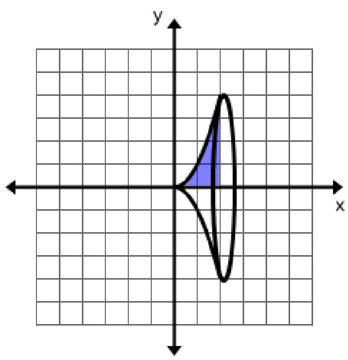
3.



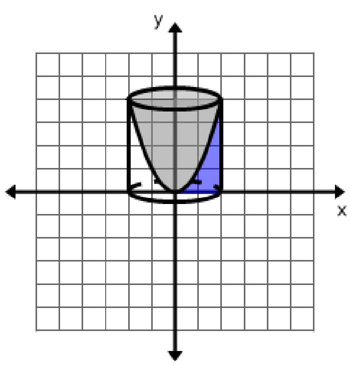
5.



7.



9.



Unit 4 Cluster 8 (G.MG.1, 2, and 3) Modeling with Geometry

Practice Exercises A (Pg. 274)

1. 31.269 in^2
3. 1222 bricks

Practice Exercises B (Pg. 276)

1. Texas has a higher population density with 97.025 people per square foot.
3. cedar plank 0.0133 density
   Oregon pine 0.0192 density
   Oregon pine is more dense

Unit 1 Cluster 2 (S.IC.1) Inferences

Practice Exercises A (Pg. 280–281)

1. a. all eligible voters in Utah
   b. all eligible voters in 15 state house districts
   c. percentage of people who vote in Utah
3. a. all Utahns over the age of 12
   b. 1200 Utahns over the age of 12
   c. the average amount of time they spend exercising
5. it is a convenience sampling and it is biased because they are conducting the survey outside of an arts program so arts supporters will be overrepresented

Unit 1 Cluster 2 (S.IC.2) Simulation

Practice Exercises A (Pg. 285)

1. Answers will vary. In 50 trials, 14 had exactly 34 students or 28% of the time.
3. Answers will vary. In 50 trials, 1 had 8 female students, which is a probability of 2%.

Unit 1 Cluster 3 (S.IC.3 and S.IC.6) Surveys, Experiments, Observations, and Evaluation of Reports

Practice Exercises A (Pg. 287)

1. survey, if the sample size is large enough the results can be applied to the population
3. experiment, if the sample size is large enough the results can be applied to the population
5. survey, if the new customers were selected randomly then the results can be applied to all new customers

Practice Exercises B (Pg. 289)

1. Answers will vary

Unit 1 Cluster 1 (S.ID.4) Normal Distribution

Practice Exercises A (Pg. 292)

1. a. 200 to 800
   b. approximately 6.7%
   c. approximately 83.5%
3. a. 71.22 inches to 86.78 inches
   b. approximately 30.6%
   c. approximately 24.2%

Unit 1 Cluster 3 (S.IC.4) Margin of Error

Practice Exercises A (Pg. 297)

1. a. sample proportion: 0.225
   margin of error: 0.132
   confidence interval: 0.093 to 0.357
   the theoretical probability of 0.25 would be in the confidence interval
   b. sample proportion: 0.125
   margin of error: 0.105
   confidence interval: 0.02 to 0.23
   the theoretical probability of 0.25 would not be in the confidence interval
   c. sample proportion: 0.35
   margin of error: 0.151
   confidence interval: 0.199 to 0.501
   the theoretical probability of 0.25 would be in the confidence interval
3. a. 52%
   b. 3.9%
   c. 48.1% to 55.9%
   d. It is plausible that the candidate has less than 50% of the vote. It is also plausible that the candidate has more than 50% of the vote so the election is too close to call.

Unit 1 Cluster 3 (S.IC.5) Compare Two Treatments

Practice Exercises A (Pg. 301)

1. Answers will vary.
   Example: the difference between the two proportions is 0.278. After running a simulation the sample proportion difference is -0.008 and the standard deviation is 0.117. The interval containing 95% of the data is -0.242 to 0.226. The sample difference is outside of this interval so it is statistically significant.